

# One Eilenberg Theorem to Rule Them All

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## Abstract

Eilenberg-type correspondences, relating varieties of languages (e.g. of finite words, infinite words, or trees) to pseudovarieties of finite algebras, form the backbone of algebraic language theory. Numerous such correspondences are known in the literature. We demonstrate that they all arise from the same recipe: one models languages and the algebras recognizing them by monads on an algebraic category, and applies a Stone-type duality. Our main contribution is a generic variety theorem that covers e.g. Wilke’s and Pin’s work on  $\infty$ -languages, the variety theorem for cost functions of Daviaud, Kuperberg, and Pin, and unifies the two previous categorical approaches of Bojańczyk and of Adámek et al. In addition it gives a number of new results, such as an extension of the local variety theorem of Gehrke, Grigorieff, and Pin from finite to infinite words.

## 1 Introduction

Algebraic language theory investigates the behaviors of finite machines by relating them to finite algebraic structures. The algebraic approach has proved very fruitful. For example, regular languages are precisely the languages recognized by finite monoids, and the decidability of star-freeness rests on Schützenberger’s theorem [34]: a language is star-free iff it is recognized by a finite aperiodic monoid. At the heart of algebraic language theory are results establishing generic correspondences of this kind. The prototype is Eilenberg’s celebrated variety theorem [16]: it states that *varieties of languages* (classes of regular languages closed under boolean operations, derivatives, and homomorphic preimages) and *pseudovarieties of monoids* (classes of finite monoids closed under quotients, submonoids, and finite products) are in bijective correspondence. This together with Reiterman’s theorem [30] (see also Banaschewski [6]), stating that pseudovarieties of monoids can be specified by *profinite equations*, establishes a firm connection between automata, languages, and algebras.

In the past decades numerous further Eilenberg-type theorems have been discovered for regular languages [18, 24, 28, 35], dealing with classes of regular languages with weaker closure properties, but also for machine behaviors beyond finite words, including weighted languages over a field [31], infinite words [25, 36], words on linear orderings [8, 9], ranked trees [5], binary trees [33], and very recently, cost functions [15]. This plethora of similar results has raised interest in generic approaches to algebraic language theory which allow to derive all the above results as special instances of only *one* general variety theorem (that therefore *rules them all*). An important first step in this direction was done by Bojańczyk [12]. He extends the classical notion of language recognition by monoids to algebras for a *monad* on sorted sets, and presents an Eilenberg theorem at this level of generality. Our previous work in [1–3, 14] takes an orthogonal approach: one keeps monoids but considers them in

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categories  $\mathcal{D}$  of (ordered) algebras such as posets, semilattices, and vector spaces. In this way we uniformly covered five Eilenberg theorems for languages of finite words [16, 24, 28, 31, 35].

In order to obtain the *one* Eilenberg theorem, a unification of the two approaches is required. On the one hand, one needs to take the step from sets to more general categories  $\mathcal{D}$  to capture the proper notion of language recognition; e.g. for the treatment of weighted languages [31] one needs to work over the category of vector spaces. On the other hand, from Bojańczyk’s work one learns that to deal with machine behaviors beyond finite words, one has to replace monoids by other algebraic structures. The main contribution of this paper is a variety theorem that achieves the desired unification, and in addition encompasses many Eilenberg-type correspondences captured by neither of the previous generic results, including the work [8, 9, 15, 25, 33, 36] mentioned above. Thus, we hope to convince the reader that our result is indeed the *one* Eilenberg theorem. Our approach starts with the observation that all Eilenberg theorems in the literature emerge essentially from the same four steps:

1. Figure out an algebraic theory such that the languages in mind are exactly the ones recognized by finite algebras. For example, for regular languages one takes monoids.
2. Find a “pretty” presentation of the finite algebras in terms of unary operations. For example, monoids can be presented by left and right multiplication with fixed elements.
3. Infer from that presentation the form of the syntactic algebras, i.e. the minimal recognizers of languages, and the type of derivatives under which varieties of languages are closed.
4. Establish a bijective correspondence between varieties of languages and pseudovarieties of algebras by relating languages to their syntactic algebras.

It turns out that all these steps can be facilitated or even completely automatized.

For Step 1, putting a common roof over Bojańczyk’s and our own previous work, we consider a locally finite variety  $\mathcal{D}$  and algebras for a *monad*  $\mathbf{T}$  on  $\mathcal{D}^S$ , the category of  $S$ -sorted  $\mathcal{D}$ -algebras for some finite set  $S$  of sorts. For example, to capture regular languages one takes the monad  $\mathbf{T}\Sigma = \Sigma^*$  on **Set** representing monoids. For regular  $\infty$ -languages one takes the monad  $\mathbf{T}(\Sigma, \Gamma) = (\Sigma^+, \Sigma^\omega + \Gamma)$  on **Set**<sup>2</sup> representing  $\omega$ -semigroups. Although, in concrete instances, one still has to find the proper variety  $\mathcal{D}$  and the monad  $\mathbf{T}$ , the search is facilitated by the fact that the free algebras for  $\mathbf{T}$  need to be the carriers of the desired languages.

For Step 2, Bojańczyk gave a generic unary presentation for any monad on sorted sets, extending work of Almeida [5]. However, for practical purposes this presentation is often too unwieldy. For example, in the case of monoids it contains all unary operations associated to words with one variable, but one wants to restrict to words where the variable appears only *once*. Therefore we make our setting parametric in a *choice* of a unary presentation of  $\mathbf{T}$ .

We emphasize that non-trivial work still lies in proving that the languages of interest are precisely those recognized by finite  $\mathbf{T}$ -algebras, and in finding a good unary presentation of  $\mathbf{T}$ . However, our work here shows that then the Steps 3 and 4 are completely generic: after choosing a unary presentation, the syntactic algebras (Theorem 3.9) and the variety theorem (Theorem 5.9) come “for free”. In fact, Theorem 3.9 even shows that a unary presentation is necessary and sufficient for constructing syntactic algebras. Our main result is the following

► **Variety Theorem.** *Varieties of languages recognizable by finite  $\mathbf{T}$ -algebras are in bijective correspondence with pseudovarieties of  $\mathbf{T}$ -algebras.*

The proof relies on two main ingredients. The first one is *duality*: besides  $\mathcal{D}$  we also consider a locally finite variety  $\mathcal{C}$  that is dual to  $\mathcal{D}$  on the level of *finite* algebras, with Stone duality ( $\mathcal{C}$  = boolean algebras and  $\mathcal{D}$  = sets) as a leading example. Varieties of languages live in  $\mathcal{C}$ , while over  $\mathcal{D}^S$  we form pseudovarieties of  $\mathbf{T}$ -algebras. Our second ingredient is the *profinite monad* of  $\mathbf{T}$ , introduced in [13]. It is inspired by the classical construction of the free profinite

monoid, and allows for the introduction of topological methods to our categorical setting. For example, Pippenger’s result [27] that the boolean algebra of regular languages is the Stone dual of the free profinite monoid is generalized to the level of monads (Theorem 3.3).

Together with our generalization of Reiterman’s theorem in [13], showing that pseudovarieties of  $\mathbf{T}$ -algebras are presentable by *profinite equations*, the variety theorem establishes a conceptual and highly parametric framework for algebraic language theory. To illustrate its strength, we demonstrate in Section 6 that it instantiates to roughly a dozen Eilenberg correspondences known in the literature. In addition, it yields new results, e.g. an extension of the local variety theorem of Gehrke, Grigorieff, and Pin [18] from finite to infinite words.

All omitted proofs and details can be found in the appendix.

## 2 The Profinite Monad

We start by introducing our categorical framework for algebraic language theory. Readers are assumed to be familiar with basic concepts from category theory such as monads and their algebras, limits, and duality [22]. The appendix contains a brief categorical toolkit.

► **Assumptions 2.1.** Throughout this paper let  $\mathcal{C}$  and  $\mathcal{D}$  be two locally finite varieties of algebras, i.e. all finitely generated algebras are finite. We assume that (i) the full subcategories  $\mathcal{C}_f$  and  $\mathcal{D}_f$  on finite algebras are dually equivalent, (ii) the signature of  $\mathcal{C}$  contains a constant, and (iii) all epimorphisms in  $\mathcal{D}$  are surjective. Finally, fix a *finite* set  $S$  of sorts and a monad  $\mathbf{T} = (T, \eta, \mu)$  on the product category  $\mathcal{D}^S$  with  $T$  preserving epimorphisms.

► **Example 2.2.** The following locally finite varieties  $\mathcal{C}$  and  $\mathcal{D}$  satisfy our assumptions:

1.  $\mathcal{C} = \mathbf{BA}$  (boolean algebras) and  $\mathcal{D} = \mathbf{Set}$ : Stone duality [19] yields a dual equivalence  $\mathbf{BA}_f^{op} \simeq \mathbf{Set}_f$ , mapping a finite boolean algebra to the set of its atoms.
2.  $\mathcal{C} = \mathcal{D} = \mathbf{JSL}_0$  (join-semilattices with 0): there is a self-duality of  $(\mathbf{JSL}_0)_f$  mapping a finite semilattice  $(X, \vee)$  to its opposite semilattice  $(X, \wedge)$ .
3.  $\mathcal{C} = \mathcal{D} = \mathbf{Vec}_K$  (vector spaces over a finite field  $K$ ): the familiar self-duality of finite (-dimensional) vector spaces maps a finite space  $X$  to its dual space  $X^* = \mathbf{Vec}_K(X, K)$ .

► **Example 2.3.** The categories  $\mathcal{C} = \mathbf{DL}_{01}$  (distributive lattices with 0 and 1) and  $\mathcal{D} = \mathbf{Pos}$  (posets) “almost” satisfy our assumptions: Birkhoff duality [10] gives a dual equivalence  $(\mathbf{DL}_{01})_f^{op} \simeq \mathbf{Pos}_f$ , mapping a finite distributive lattice to the poset of its join-irreducible elements, but  $\mathbf{Pos}$  is not a variety of algebras. However, we will later see that our setting extends to varieties  $\mathcal{D}$  of *ordered* algebras (including  $\mathbf{Pos}$ ), see Remark 3.12.

► **Example 2.4.** Our monads  $\mathbf{T}$  of interest represent structures in formal language theory.

1. Let  $\mathbf{T}_*$  be the free-monoid monad on  $\mathbf{Set}$ . Languages of finite words correspond to subsets of  $T_*\Sigma = \Sigma^*$ . The category of  $\mathbf{T}_*$ -algebras is isomorphic to the category of monoids.
2. Languages of finite and infinite words (i.e.  $\omega$ -languages) are represented by the monad  $\mathbf{T}_\infty$  on  $\mathbf{Set}^2$  associated to the algebraic theory of  $\omega$ -semigroups. Recall that an  $\omega$ -semigroup is a two-sorted set  $A = (A_+, A_\omega)$  equipped with a binary product  $A_+ \times A_+ \rightarrow A_+$ , a mixed binary product  $A_+ \times A_\omega \rightarrow A_\omega$  and an  $\omega$ -ary product  $A_+^\omega \rightarrow A_\omega$  satisfying all (mixed) associative laws [23]. The free  $\omega$ -semigroup on the two-sorted set  $(\Sigma, \Gamma)$  is  $(\Sigma^+, \Sigma^\omega + \Gamma)$  with products given by concatenation. Thus  $T_\infty(\Sigma, \Gamma) = (\Sigma^+, \Sigma^\omega + \Gamma)$ , and an  $\omega$ -language over the alphabet  $\Sigma$  corresponds to a two-sorted subset of  $T_\infty(\Sigma, \emptyset) = (\Sigma^+, \Sigma^\omega)$ .
3. Weighted languages  $L: \Sigma^* \rightarrow K$  over a finite field  $K$  are represented by the monad  $\mathbf{T}_K$  on  $\mathbf{Vec}_K$  constructing free  $K$ -algebras. For the vector space  $K^\Sigma$  with finite basis  $\Sigma$  we have  $T_K(K^\Sigma) = K[\Sigma]$ , i.e. polynomials  $\sum_{i < n} k_i w_i$  with  $k_i \in K$  and  $w_i \in \Sigma^*$ . Since  $K[\Sigma]$  has the basis  $\Sigma^*$ , weighted languages correspond to linear maps from  $T_K(K^\Sigma)$  to  $K$ .

► **Remark 2.5.** The variety  $\mathcal{D}$  has the factorization system of surjective morphisms (= epimorphisms) and injective morphisms, extending sortwise to  $\mathcal{D}^S$ . Denote by  $\mathbf{Alg}_f \mathbf{T}$  and  $\mathbf{Alg} \mathbf{T}$  the categories of (finite)  $\mathbf{T}$ -algebras and  $\mathbf{T}$ -homomorphisms. Since  $T$  preserves epimorphisms, the factorization system of  $\mathcal{D}^S$  lifts to  $\mathbf{Alg} \mathbf{T}$ : every  $\mathbf{T}$ -homomorphism factorizes into a (sortwise) surjective homomorphism followed by an injective one. *Quotients* and *subalgebras* in  $\mathbf{Alg} \mathbf{T}$  are taken in this factorization system.

A fundamental tool for describing varieties of languages and pseudovarieties of monoids are the *profinite words* over an alphabet  $\Sigma$ . They form the Stone space  $\widehat{\Sigma}^*$  arising as the inverse (a.k.a. cofiltered) limit of all finite quotient monoids of  $\Sigma^*$ . In [13] we generalized this construction from the free-monoid monad  $\mathbf{T}_*$  on  $\mathbf{Set}$  to arbitrary monads  $\mathbf{T}$  as follows.

► **Notation 2.6.** Let  $\mathbf{Stone}(\mathcal{D})$  be the category of topological  $\mathcal{D}$ -algebras carrying a Stone topology, and continuous  $\mathcal{D}$ -morphisms. Denote by  $\widehat{\mathcal{D}}$  the full subcategory of  $\mathbf{Stone}(\mathcal{D})$  on *profinite  $\mathcal{D}$ -algebras*, i.e. inverse limits of algebras in  $\mathcal{D}_f$ . Note that  $\mathcal{D}_f$  is a full subcategory of  $\widehat{\mathcal{D}}$ , by identifying objects of  $\mathcal{D}_f$  with profinite  $\mathcal{D}$ -algebras carrying a discrete topology.

► **Example 2.7.** We have  $\widehat{\mathbf{Set}} = \mathbf{Stone}$ ,  $\widehat{\mathbf{JSL}_0} = \mathbf{Stone}(\mathbf{JSL}_0)$  and  $\widehat{\mathbf{Vec}_K} = \mathbf{Stone}(\mathbf{Vec}_K)$ . Thus, in these examples all Stone algebras are profinite, see [19].

► **Construction 2.8** (see [13]). For an object  $D \in \mathcal{D}_f^S$  form the poset  $\mathbf{Quo}_f(\mathbf{T}D)$  of finite quotient algebras  $e: \mathbf{T}D \twoheadrightarrow (A, \alpha)$  of the free  $\mathbf{T}$ -algebra  $\mathbf{T}D = (TD, \mu_D)$ , ordered by  $e \leq e'$  iff  $e$  factors through  $e'$ . Define  $\widehat{T}D$  in  $\widehat{\mathcal{D}}^S$  to be the inverse limit of the diagram  $\mathbf{Quo}_f(\mathbf{T}D) \rightarrow \widehat{\mathcal{D}}^S$  mapping  $(e: \mathbf{T}D \twoheadrightarrow (A, \alpha))$  to  $A$ . We denote the limit projections by  $e^+: \widehat{T}D \twoheadrightarrow A$ .

► **Theorem 2.9** (see [13]). *The object map  $D \mapsto \widehat{T}D$  from  $\mathcal{D}_f^S$  to  $\widehat{\mathcal{D}}^S$  extends (via inverse limits) to a functor  $\widehat{T}: \widehat{\mathcal{D}}^S \rightarrow \widehat{\mathcal{D}}^S$ . Further,  $\widehat{T}$  can be equipped with the structure of a monad  $\widehat{\mathbf{T}} = (\widehat{T}, \widehat{\eta}, \widehat{\mu})$  called the profinite monad of  $\mathbf{T}$ . Its unit  $\widehat{\eta}_D$  and multiplication  $\widehat{\mu}_D$  for  $D \in \mathcal{D}_f^S$  are determined by the commutative diagrams below for all  $e: \mathbf{T}D \twoheadrightarrow (A, \alpha)$  in  $\mathbf{Quo}_f(\mathbf{T}D)$ :*

$$\begin{array}{ccc}
 D & \xrightarrow{\widehat{\eta}_D} & \widehat{T}D \\
 & \searrow e\eta_D & \downarrow e^+ \\
 & & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 \widehat{T}\widehat{T}D & \xrightarrow{\widehat{\mu}_D} & \widehat{T}D \\
 \widehat{T}e^+ \downarrow & & \downarrow e^+ \\
 \widehat{T}A & \xrightarrow{\alpha^+} & A
 \end{array}
 \tag{1}$$

Here the limit projection  $\alpha^+$  exists since  $\alpha: \mathbf{T}A \twoheadrightarrow (A, \alpha)$  is a surjective  $\mathbf{T}$ -homomorphism.

► **Example 2.10.**  $\widehat{\mathbf{T}}_*$  is the monad on  $\mathbf{Stone}$  assigning to each finite set  $\Sigma$  the space  $\widehat{\Sigma}^*$  of profinite words. Similarly,  $\widehat{\mathbf{T}}_K$  is the monad on  $\mathbf{Stone}(\mathbf{Vec}_K)$  assigning to each finite vector space  $K^\Sigma$  the Stone vector space obtained as the limit of all finite quotient spaces of  $K[\Sigma]$ .

- **Remark 2.11.** 1. If  $(A, \alpha)$  is a finite  $\mathbf{T}$ -algebra, then  $(A, \alpha^+)$  is a finite  $\widehat{\mathbf{T}}$ -algebra: the unit and associative law follow from (1) with  $D = A$  and  $e = \alpha$ . By [13, Proposition 3.10] this yields an isomorphism  $\mathbf{Alg}_f \mathbf{T} \cong \mathbf{Alg}_f \widehat{\mathbf{T}}$  given by  $(A, \alpha) \mapsto (A, \alpha^+)$  and  $h \mapsto h$ .
2. Let  $V$  denote the two forgetful functors  $V: \widehat{\mathcal{D}} \rightarrow \mathcal{D}$  and  $V: \widehat{\mathcal{D}}^S \rightarrow \mathcal{D}^S$ . By [13, Remark B.6] there is a natural transformation  $\iota: TV \rightarrow V\widehat{T}$  whose component  $\iota_D: TVD \rightarrow V\widehat{T}D$  for  $D \in \mathcal{D}_f^S$  satisfies  $V e^+ \cdot \iota_D = e$  for all finite quotient algebras  $e: \mathbf{T}D \twoheadrightarrow A$  of  $\mathbf{T}D$  in  $\mathbf{Alg} \mathbf{T}$ . More generally, we call a finite quotient  $e: \mathbf{T}D \twoheadrightarrow A$  in  $\mathcal{D}^S$  *extensible* if  $V \widehat{e} \cdot \iota_D = e$  for some  $\widehat{e}: \widehat{T}D \twoheadrightarrow A$  in  $\widehat{\mathcal{D}}^S$ . Thus every finite quotient of  $\mathbf{T}D$  in  $\mathbf{Alg} \mathbf{T}$  is extensible.

- **Remark 2.12.** 1. The category  $\widehat{\mathcal{D}}$  is the *pro-completion* (the free completion under inverse limits) of  $\mathcal{D}_f$ , see [19, Remark VI.2.4]. Moreover, since  $\mathcal{C}$  is locally finite,  $\mathcal{C}$  is the *ind-completion* (the free completion under filtered colimits) of  $\mathcal{C}_f$ . Thus the dual equivalence between  $\mathcal{C}_f$  and  $\mathcal{D}_f$  extends to a dual equivalence between  $\mathcal{C}$  and  $\widehat{\mathcal{D}}$ . By  $P: \widehat{\mathcal{D}} \xrightarrow{\sim} \mathcal{C}^{op}$  and  $P^{-1}: \mathcal{C}^{op} \xrightarrow{\sim} \widehat{\mathcal{D}}$  denote the equivalence functors. For  $\mathcal{C} = \mathbf{BA}$  and  $\mathcal{D} = \mathbf{Set}$  (with  $\widehat{\mathcal{D}} = \mathbf{Stone}$ ), this is the classical Stone duality [19]:  $P$  maps a Stone space to the boolean algebra of clopens, and  $P^{-1}$  maps a boolean algebra to the Stone space of all ultrafilters.
2. Let  $\mathbb{1}$  denote the free one-generated objects both in  $\mathcal{C}$  and  $\widehat{\mathcal{D}}$  (w.r.t. the forgetful functors  $|-|: \mathcal{C} \rightarrow \mathbf{Set}$  and  $|-|: \widehat{\mathcal{D}} \rightarrow \mathbf{Set}$ ). The two finite objects  $O_{\mathcal{C}} := P\mathbb{1}$  and  $O_{\mathcal{D}} := P^{-1}\mathbb{1}$  play the role of a *dualizing object* (also called a *schizophrenic object* in [19]) of  $\mathcal{C}$  and  $\widehat{\mathcal{D}}$ . This means that there is a natural isomorphism  $|P| \cong \widehat{\mathcal{D}}(-, O_{\mathcal{D}})$  given by

$$|PD| \cong \mathcal{C}(\mathbb{1}, PD) \cong \widehat{\mathcal{D}}(P^{-1}PD, O_{\mathcal{D}}) \cong \widehat{\mathcal{D}}(D, O_{\mathcal{D}})$$

for  $D \in \widehat{\mathcal{D}}$ , and analogously an isomorphism  $|P^{-1}| \cong \mathcal{C}(-, O_{\mathcal{C}})$ . In particular, we have

$$|O_{\mathcal{D}}| \cong \widehat{\mathcal{D}}(\mathbb{1}, O_{\mathcal{D}}) \cong |P\mathbb{1}| = |O_{\mathcal{C}}|.$$

3. *Subobjects* in the variety  $\mathcal{C}$  are represented by monomorphisms (= injective morphisms). Dually, *quotients* in  $\widehat{\mathcal{D}}$  are represented by epimorphisms. From our assumption that epimorphisms in  $\mathcal{D}$  are the surjective morphisms, one can show that the same holds in  $\widehat{\mathcal{D}}$ . *Quotients* of  $\widehat{\mathbf{T}}$ -algebras are thus represented by sortwise surjective  $\widehat{\mathbf{T}}$ -homomorphisms.

### 3 Recognizable Languages and Syntactic $\mathbf{T}$ -Algebras

A language  $L \subseteq \Sigma^*$  may be identified with its characteristic function  $L: \Sigma^* \rightarrow \{0, 1\}$ . To get a notion of *language* in our categorical setting, we replace the one-sorted alphabet  $\Sigma$  by an  $S$ -sorted alphabet  $\Sigma$  in  $\mathbf{Set}_f^S$ , and represent it in  $\mathcal{D}^S$  via the free object  $\Sigma \in \mathcal{D}_f^S$  generated by  $\Sigma$  (w.r.t. the forgetful functor  $|-|: \mathcal{D}^S \rightarrow \mathbf{Set}^S$ ). The set  $\{0, 1\}$  is replaced by a finite “object of outputs” in  $\mathcal{D}_f^S$ , viz. the object with  $O_{\mathcal{D}} \in \mathcal{D}_f$  in each sort. We denote this object of  $\mathcal{D}_f^S$  also by  $O_{\mathcal{D}}$ . This leads to the following definition, unifying concepts in [12] and [2].

► **Definition 3.1.** A *language* over  $\Sigma \in \mathbf{Set}_f^S$  is a morphism  $L: T\Sigma \rightarrow O_{\mathcal{D}}$  in  $\mathcal{D}^S$ . It is *recognized* by a  $\mathbf{T}$ -homomorphism  $h: T\Sigma \rightarrow (A, \alpha)$  if there is a morphism  $p: A \rightarrow O_{\mathcal{D}}$  in  $\mathcal{D}^S$  with  $L = p \cdot h$ . A language is  *$\mathbf{T}$ -recognizable* if it is recognized by some  $\mathbf{T}$ -homomorphism with finite codomain. We denote the set of all  $\mathbf{T}$ -recognizable languages over  $\Sigma$  by  $\text{Rec}(\Sigma)$ .

- **Example 3.2.** 1.  $\mathbf{T} = \mathbf{T}_*$  on  $\mathbf{Set}$  with  $O_{\mathbf{Set}} = \{0, 1\}$ : a language  $L: T_*\Sigma \rightarrow \{0, 1\}$  corresponds to a classical language  $L \subseteq \Sigma^*$  of finite words. It is recognized by a monoid morphism  $h: \Sigma^* \rightarrow A$  iff  $L = h^{-1}[Y]$  for some subset  $Y \subseteq A$ . Recognizable languages coincide with regular languages, i.e. languages accepted by finite automata [26].
2.  $\mathbf{T} = \mathbf{T}_{\infty}$  on  $\mathbf{Set}^2$  with  $O_{\mathbf{Set}} = \{0, 1\}$ : since  $T_{\infty}(\Sigma, \emptyset) = (\Sigma^+, \Sigma^{\omega})$ , a language  $L: T_{\infty}(\Sigma, \emptyset) \rightarrow \{0, 1\}$  corresponds to an  $\infty$ -language  $L \subseteq \Sigma^+ \cup \Sigma^{\omega}$ . It is recognized by an  $\omega$ -semigroup morphism  $h: (\Sigma^+, \Sigma^{\omega}) \rightarrow A$  iff  $L = h^{-1}[Y]$  for some  $Y \subseteq A$ . Recognizable  $\infty$ -languages coincide with regular  $\infty$ -languages, i.e. languages accepted by finite Büchi automata [23].

A key observation for the topological approach to automata theory is that regular languages over  $\Sigma$  correspond to clopen subsets of the Stone space  $\widehat{\Sigma}^*$  of profinite words, or equivalently, to continuous maps from  $\widehat{\Sigma}^*$  into the discrete space  $\{0, 1\}$ ; see e.g. [26, Proposition VI.3.12]. This result generalizes from the monad  $\mathbf{T}_*$  on  $\mathbf{Set}$  to arbitrary monads  $\mathbf{T}$ :

► **Theorem 3.3.** *Recognizable languages over  $\Sigma$  correspond uniquely to morphisms from  $\hat{T}\Sigma$  to  $O_{\mathcal{D}}$  in  $\hat{\mathcal{D}}^S$ .*

**Proof sketch.** For any recognizable language  $L: T\Sigma \rightarrow O_{\mathcal{D}}$ , choose a finite quotient algebra  $e: \mathbf{T}\Sigma \rightarrow (A, \alpha)$  and a morphism  $p: A \rightarrow O_{\mathcal{D}}$  with  $L = e \cdot p$ . This yields the morphism  $\hat{L} := p \cdot e^+: \hat{T}\Sigma \rightarrow O_{\mathcal{D}}$  in  $\hat{\mathcal{D}}^S$ , where  $e^+$  is the limit projection of Construction 2.8. Conversely, every morphism  $\hat{L}: \hat{T}\Sigma \rightarrow O_{\mathcal{D}}$  in  $\hat{\mathcal{D}}^S$  restricts to the recognizable language  $L := V\hat{L} \cdot \iota_{\Sigma}: T\Sigma \rightarrow O_{\mathcal{D}}$ , cf. Remark 2.11.2. The maps  $L \mapsto \hat{L}$  and  $\hat{L} \mapsto L$  are mutually inverse. ◀

► **Remark 3.4.** From the above theorem and Remark 2.12.2 we get

$$\text{Rec}(\Sigma) \cong \hat{\mathcal{D}}^S(\hat{T}\Sigma, O_{\mathcal{D}}) \cong \prod_s \hat{\mathcal{D}}((\hat{T}\Sigma)_s, O_{\mathcal{D}}) \cong \prod_s |P(\hat{T}\Sigma)_s|$$

Thus we can represent  $\text{Rec}(\Sigma)$  as an object of  $\mathcal{C}$  isomorphic to  $\prod_s P(\hat{T}\Sigma)_s$ . One can show that  $\text{Rec}(\Sigma)$  forms a subobject of  $\prod_s O_{\mathcal{C}}^{|T\Sigma|_s}$ : the embedding  $\text{Rec}(\Sigma) \hookrightarrow \prod_s O_{\mathcal{C}}^{|T\Sigma|_s}$  maps a language  $L: T\Sigma \rightarrow O_{\mathcal{D}}$  to the tuple  $(|T\Sigma|_s \xrightarrow{|L|} |O_{\mathcal{D}}| \xrightarrow{\cong} |O_{\mathcal{C}}|)_{s \in S}$ , using the bijection  $|O_{\mathcal{D}}| \cong |O_{\mathcal{C}}|$  of Remark 2.12.2. Consequently the  $\mathcal{C}$ -algebraic structure of  $\text{Rec}(\Sigma)$  is determined by  $O_{\mathcal{C}}$ . For example, for  $\mathcal{C} = \mathbf{BA}$  with  $O_{\mathbf{BA}} = \{0, 1\}$ , the boolean structure of  $\text{Rec}(\Sigma)$  is given by union, intersection and complement. For  $\mathbf{T} = \mathbf{T}_*$  on  $\mathbf{Set}$ , we thus recover a result of Pippenger [27]: the boolean algebra of regular languages over  $\Sigma$  is dual to the Stone space  $\widehat{\Sigma}^*$ .

An important tool for the algebraic approach to regular languages is the syntactic monoid of a language, viz. the smallest monoid recognizing it. We now introduce syntactic algebras for  $\mathbf{T}$ -recognizable languages, unifying the two corresponding concepts studied in [12] and [2].

► **Definition 3.5.** Let  $L: T\Sigma \rightarrow O_{\mathcal{D}}$  be recognizable. A *syntactic  $\mathbf{T}$ -algebra* of  $L$  is a finite  $\mathbf{T}$ -algebra  $A_L$  together with a surjective  $\mathbf{T}$ -homomorphism  $e_L: \mathbf{T}\Sigma \rightarrow A_L$  (called a *syntactic morphism* of  $L$ ) such that (i)  $e_L$  recognizes  $L$ , and (ii)  $e_L$  factors through any surjective  $\mathbf{T}$ -homomorphism  $e: \mathbf{T}\Sigma \rightarrow A$  recognizing  $L$ , i.e.  $e_L = h \cdot e$  for some  $h: A \rightarrow A_L$  in  $\mathbf{Alg} \mathbf{T}$ .

- **Example 3.6.** 1.  $\mathbf{T} = \mathbf{T}_*$  on  $\mathbf{Set}$ : the *syntactic monoid* [26] of a recognizable language  $L: \Sigma^* \rightarrow \{0, 1\}$  is the quotient monoid  $e_L: \Sigma^* \rightarrow \Sigma^*/\equiv_L$ , where  $\equiv_L$  is the monoid congruence on  $\Sigma^*$  defined by  $v \equiv_L w$  iff  $L(xvy) = L(xwy)$  for all  $x, y \in \Sigma^*$ .
2.  $\mathbf{T} = \mathbf{T}_{\infty}$  on  $\mathbf{Set}^2$ : the *syntactic  $\omega$ -semigroup* [23] of a recognizable language  $L: (\Sigma^+, \Sigma^{\omega}) \rightarrow \{0, 1\}$  is the quotient  $\omega$ -semigroup  $e_L: (\Sigma^+, \Sigma^{\omega}) \rightarrow (\Sigma^+, \Sigma^{\omega})/\equiv_L$ , where  $\equiv_L$  is the following  $\omega$ -semigroup congruence on  $(\Sigma^+, \Sigma^{\omega})$ : for  $v, w \in \Sigma^+$  put  $v \equiv_L w$  iff  $L(xvy) = L(xwy)$ ,  $L(xvz) = L(xwz)$  and  $L(x(vy)^{\omega}) = L(x(wy)^{\omega})$  for all  $x, y \in \Sigma^*$  and  $z \in \Sigma^{\omega}$ . And for  $v, w \in \Sigma^{\omega}$  put  $v \equiv_L w$  iff  $L(xv) = L(xw)$  for all  $x \in \Sigma^*$ .
3. Let  $\mathbf{T}$  be any monad on  $\mathbf{Set}^S$ . Generalizing work of Almeida [5] on algebras for a finitary signature, Bojańczyk [12] showed that every  $\mathbf{T}$ -recognizable language  $L: T\Sigma \rightarrow \{0, 1\}$  has a syntactic  $\mathbf{T}$ -algebra, constructed as follows. Denote by  $1_s \in \mathbf{Set}^S$  the  $S$ -sorted set with one element in sort  $s$  and otherwise empty; thus a morphism  $1_s \rightarrow A$  in  $\mathbf{Set}^S$  chooses an element of  $A_s$ . A *polynomial* over  $\Sigma$  is a morphism  $p: 1_{s'} \rightarrow T(\Sigma + 1_s)$  with  $s, s' \in S$ , i.e. a “term” of output sort  $s'$  in a variable of sort  $s$ . Every polynomial induces an evaluation map  $(T\Sigma)_s \xrightarrow{[p]} (T\Sigma)_{s'}$  that inserts elements of  $(T\Sigma)_s$  for the variable. The syntactic  $\mathbf{T}$ -algebra of  $L$  is given by  $e_L: \mathbf{T}\Sigma \rightarrow \mathbf{T}\Sigma/\equiv_L$ , where  $\equiv_L$  is defined on sort  $s$  by  $x \equiv_L y$  iff  $L \cdot [p](x) = L \cdot [p](y)$  for all polynomials  $p: 1_{s'} \rightarrow T(\Sigma + 1_s)$  with  $s' \in S$ .

In each of the above examples,  $\equiv_L$  is based on certain unary operations. For monoids one uses the operations  $v \mapsto xvy$  on  $\Sigma^*$ . They determine the syntactic morphism as the monoid structure of any quotient of  $\Sigma^*$  can be recovered from these operations. For  $\omega$ -semigroups,



$\equiv_L$  uses the operations  $v \mapsto xvy$  on  $\Sigma^+$ ,  $v \mapsto xvz$  from  $\Sigma^+$  to  $\Sigma^\omega$ ,  $v \mapsto x(vy)^\omega$  from  $\Sigma^+$  to  $\Sigma^\omega$ , and  $v \mapsto xv$  on  $\Sigma^\omega$ . They determine any finite  $\omega$ -semigroup, see Wilke [36]. In the last example, the operations are  $(T\Sigma)_s \xrightarrow{[p]} (T\Sigma)_{s'}$ , and again this works as any finite quotient of  $T\Sigma$  is determined by the polynomials. Here is a categorical formulation of this phenomenon:

► **Definition 3.7.** Let  $\Sigma \in \mathbf{Set}_f^S$ . A *unary operation on  $T\Sigma$*  is a morphism  $u: (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathcal{D}$ , where  $s$  and  $s'$  are arbitrary sorts. A set  $\mathbb{U}_\Sigma$  of unary operations on  $T\Sigma$  is called a *unary presentation of  $\mathbf{T}$  over  $\Sigma$*  if for any extensible finite quotient  $e: T\Sigma \twoheadrightarrow A$  in  $\mathcal{D}^S$  (see Remark 2.11.2) the following are equivalent:

- (i)  $e$  is a  *$\mathbf{T}$ -algebra congruence* on  $T\Sigma$ , i.e. there exists a  $\mathbf{T}$ -algebra structure  $(A, \alpha)$  on  $A$  for which  $e: T\Sigma \twoheadrightarrow (A, \alpha)$  is a  $\mathbf{T}$ -homomorphism.
- (ii) Each operation  $u: (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbb{U}_\Sigma$  has a lifting along  $e$ , i.e. there exists a morphism  $u_A: A_s \rightarrow A_{s'}$  in  $\mathcal{D}$  with  $e \cdot u = u_A \cdot e$ .

► **Definition 3.8.** Let  $\mathbb{U}_\Sigma$  be a set of unary operations on  $T\Sigma$ . The *syntactic equivalence* of a language  $L: T\Sigma \rightarrow O_{\mathcal{D}}$  (w.r.t.  $\mathbb{U}_\Sigma$ ) is the  $S$ -sorted equivalence relation  $\equiv_L$  on  $|T\Sigma|$  defined as follows: for elements  $x, y \in |T\Sigma|_s$  put

$$x \equiv_L y \quad \text{iff} \quad L \cdot u(x) = L \cdot u(y) \quad \text{for all sorts } s' \text{ and all } u: (T\Sigma)_s \rightarrow (T\Sigma)_{s'} \text{ in } \mathbb{U}_\Sigma.$$

Observe that  $\equiv_L$  is a *congruence* on  $T\Sigma$  in  $\mathcal{D}^S$ , being the intersection of the kernel congruences  $K_{s',u} = \{ (x, y) : L \cdot u(x) = L \cdot u(y) \}$  with  $s' \in S$  and  $u: (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbb{U}_\Sigma$ . Hence one can form the quotient  $e_L: T\Sigma \twoheadrightarrow T\Sigma / \equiv_L$  in  $\mathcal{D}^S$ , and it is natural to ask when  $T\Sigma / \equiv_L$  carries a syntactic  $\mathbf{T}$ -algebra for  $L$ . This turns out to hold whenever  $\mathbb{U}_\Sigma$  is a unary presentation:

► **Theorem 3.9.** Let  $\mathbb{U}_\Sigma$  be a set of unary operations on  $T\Sigma$  closed under composition and containing all identity morphisms  $\text{id}: (T\Sigma)_s \rightarrow (T\Sigma)_s$ . Then  $\mathbb{U}_\Sigma$  forms a unary presentation of  $\mathbf{T}$  over  $\Sigma$  if and only if for each recognizable language  $L: T\Sigma \rightarrow O_{\mathcal{D}}$  the morphism  $e_L: T\Sigma \twoheadrightarrow T\Sigma / \equiv_L$  is a  $\mathbf{T}$ -algebra congruence on  $T\Sigma$ , and  $e_L$  is a syntactic morphism of  $L$ .

► **Remark 3.10.** A set  $\mathbb{U}_\Sigma$  of unary operations on  $T\Sigma$  is a unary presentation iff its closure under composition and identities is. Therefore the “if” part of the theorem holds for any  $\mathbb{U}_\Sigma$ .

- **Example 3.11.** 1.  $\mathbf{T} = \mathbf{T}_*$  on  $\mathbf{Set}$ : by Example 3.6.1 and Theorem 3.9, we have for all  $\Sigma \in \mathbf{Set}_f$  the unary presentation  $\mathbb{U}_\Sigma = \{ \Sigma^* \xrightarrow{x \cdot - \cdot y} \Sigma^* : x, y \in \Sigma^* \}$ .
2.  $\mathbf{T} = \mathbf{T}_\infty$  on  $\mathbf{Set}^2$ : by Example 3.6.2 and Theorem 3.9, we have for all  $\bar{\Sigma} = (\Sigma, \emptyset) \in \mathbf{Set}_f^2$  the unary presentation  $\mathbb{U}_{\bar{\Sigma}}$  consisting of the maps  $\Sigma^+ \xrightarrow{x \cdot - \cdot y} \Sigma^+$ ,  $\Sigma^+ \xrightarrow{x \cdot - \cdot z} \Sigma^\omega$ ,  $\Sigma^+ \xrightarrow{x \cdot (-y)^\omega} \Sigma^\omega$  and  $\Sigma^\omega \xrightarrow{x \cdot -} \Sigma^\omega$  with  $x, y \in \Sigma^*$  and  $z \in \Sigma^\omega$ .
3. Let  $\mathbf{T}$  be any monad on  $\mathbf{Set}^S$ . By Example 3.6.3 and Theorem 3.9 we have for all  $\Sigma \in \mathbf{Set}_f^S$  the unary presentation  $\mathbb{U}_\Sigma = \{ (T\Sigma)_s \xrightarrow{[p]} (T\Sigma)_{s'} : p \text{ is a polynomial over } \Sigma \}$ .

► **Remark 3.12.** All our results extend to varieties  $\mathcal{D}$  of *ordered algebras* [11], cf. Example 2.3. Here Stone spaces are replaced by *Priestley spaces* [29], i.e. ordered compact spaces where any two elements  $x \not\leq y$  can be separated by clopen upper set, and  $\mathbf{Stone}(\mathcal{D})$  is replaced by the category  $\mathbf{Priest}(\mathcal{D})$  of ordered topological  $\mathcal{D}$ -algebras with a Priestley topology.  $\widehat{\mathcal{D}}$  is the full subcategory of  $\mathbf{Priest}(\mathcal{D})$  on profinite  $\mathcal{D}$ -algebras. The construction of  $\widehat{\mathbf{T}}$  is as above. In lieu of the syntactic equivalence in Definition 3.8 one uses the *syntactic preorder*

$$x \leq_L y \quad \text{iff} \quad L \cdot u(x) \leq L \cdot u(y) \quad \text{for all sorts } s' \text{ and all } u: (T\Sigma)_s \rightarrow (T\Sigma)_{s'} \text{ in } \mathbb{U}_\Sigma,$$

and forms the induced poset  $T\Sigma / \leq_L$ . For the “only if” part of Theorem 3.9 one needs to assume that  $O_{\mathcal{D}}$  is an *order-cogenerator*, i.e. for any two elements  $x \not\leq y$  of  $D \in \mathcal{D}_f$  there is a morphism  $k: D \rightarrow O_{\mathcal{D}}$  with  $k(x) \not\leq k(y)$ . This holds e.g. for  $\mathcal{D} = \mathbf{Pos}$  with  $O_{\mathbf{Pos}} = \{0 < 1\}$ .

#### 4 Pseudovarieties of $\mathbf{T}$ -algebras

In this section we introduce pseudovarieties of  $\mathbf{T}$ -algebras, the algebraic half of any Eilenberg-type correspondence, and investigate their connection to profinite  $\widehat{\mathbf{T}}$ -algebras.

► **Definition 4.1.** A  $\Sigma$ -generated finite  $\mathbf{T}$ -algebra is a finite quotient  $e: \mathbf{T}\Sigma \twoheadrightarrow A$  of  $\mathbf{T}\Sigma$  in  $\mathbf{Alg} \mathbf{T}$ . The *subdirect product* of  $e_i: \mathbf{T}\Sigma \twoheadrightarrow A_i$  ( $i = 0, 1$ ) is the image  $e: \mathbf{T}\Sigma \twoheadrightarrow A$  of the  $\mathbf{T}$ -homomorphism  $\langle e_0, e_1 \rangle: \mathbf{T}\Sigma \rightarrow A_0 \times A_1$ . We say that  $e_1$  is a *quotient* of  $e_0$  iff  $e_1$  factors through  $e_0$ . By a *local pseudovariety of  $\Sigma$ -generated  $\mathbf{T}$ -algebras* is meant a class of  $\Sigma$ -generated finite  $\mathbf{T}$ -algebras closed under subdirect products and quotients.

► **Definition 4.2.** A  $\widehat{\mathbf{T}}$ -algebra is *profinite* if it is an inverse limit of finite  $\widehat{\mathbf{T}}$ -algebras (cf. Remark 2.11.1). A  $\Sigma$ -generated profinite  $\widehat{\mathbf{T}}$ -algebra is a profinite quotient  $e: \widehat{\mathbf{T}}\Sigma \twoheadrightarrow A$  of  $\widehat{\mathbf{T}}\Sigma$  in  $\mathbf{Alg} \widehat{\mathbf{T}}$ .  $\Sigma$ -generated profinite  $\widehat{\mathbf{T}}$ -algebras are ordered by  $e \leq e'$  iff  $e$  factors through  $e'$ .

► **Proposition 4.3.** For each  $\Sigma \in \mathbf{Set}_f^S$ , the poset of local pseudovarieties of  $\Sigma$ -generated  $\mathbf{T}$ -algebras (ordered by inclusion) is isomorphic to the poset of  $\Sigma$ -generated profinite  $\widehat{\mathbf{T}}$ -algebras.

**Proof sketch.** For any  $\Sigma$ -generated profinite  $\widehat{\mathbf{T}}$ -algebra  $e: \widehat{\mathbf{T}}\Sigma \twoheadrightarrow A$ , form the class  $\mathcal{P}^e$  of all  $\Sigma$ -generated finite  $\mathbf{T}$ -algebras arising as quotients of  $A$  (cf. Remark 2.11.1). This is easily seen to be a local pseudovariety, and the map  $e \mapsto \mathcal{P}^e$  gives the isomorphism. ◀

► **Remark 4.4.** Proposition 4.3 has an equational interpretation. A *profinite equation over  $\Sigma$*  is a pair of elements  $u, v \in |\widehat{\mathbf{T}}\Sigma|_s$  in some sort  $s$ . We say that a  $\Sigma$ -generated finite  $\mathbf{T}$ -algebra  $e: \mathbf{T}\Sigma \twoheadrightarrow A$  *satisfies the equation  $u = v$*  if  $e^+(u) = e^+(v)$ . Local pseudovarieties are precisely the classes of  $\Sigma$ -generated finite  $\mathbf{T}$ -algebras presentable by profinite equations over  $\Sigma$ .

Eilenberg's variety theorem deals, in lieu of languages over a fixed alphabet, with all alphabets at once. We will do the same in all our one-sorted applications. However, as suggested by Example 2.4.2, in a many-sorted setting one often needs to make a suitable *choice* of alphabets in  $\mathbf{Set}_f^S$ . Therefore, for the rest of this paper, we fix a class  $\mathbb{A} \subseteq \mathbf{Set}_f^S$  of alphabets.

► **Definition 4.5.** A  $\mathbf{T}$ -algebra  $A$  is  $\mathbb{A}$ -generated if there exists a surjective  $\mathbf{T}$ -homomorphism  $e: \mathbf{T}\Sigma \twoheadrightarrow A$  for some  $\Sigma \in \mathbb{A}$ . By a *pseudovariety of  $\mathbf{T}$ -algebras* is meant a class of  $\mathbb{A}$ -generated finite  $\mathbf{T}$ -algebras closed under quotients and  $\mathbb{A}$ -generated subalgebras of finite products.

► **Remark 4.6.** In most applications all finite products of  $\mathbb{A}$ -generated  $\mathbf{T}$ -algebras are  $\mathbb{A}$ -generated. In this case the definition of a pseudovariety simplifies: it is a class of  $\mathbb{A}$ -generated finite  $\mathbf{T}$ -algebras closed under quotients,  $\mathbb{A}$ -generated subalgebras, and finite products.

- **Example 4.7.** 1. Every finite  $\mathbf{T}$ -algebra  $(A, \alpha)$  is  $\mathbf{Set}_f^S$ -generated: since  $\mathcal{D}$  is locally finite, there exists an epimorphism  $e: \Sigma \twoheadrightarrow A$  with  $\Sigma \in \mathbf{Set}_f^S$ , so  $(A, \alpha)$  is a quotient of  $\mathbf{T}\Sigma$  via  $(\mathbf{T}\Sigma \xrightarrow{T_e} \mathbf{T}A \xrightarrow{\alpha} (A, \alpha))$ . Thus, for  $\mathbb{A} = \mathbf{Set}_f^S$ , a pseudovariety of  $\mathbf{T}$ -algebras is a class of finite  $\mathbf{T}$ -algebras closed under quotients, subalgebras, and finite products. This concept was studied in [13]. For the monad  $\mathbf{T}_*$  on  $\mathbf{Set}$  we get the original concept of Eilenberg: a class of finite monoids closed under quotients, submonoids, and finite products.
2. Let  $\mathbf{T} = \mathbf{T}_\infty$  on  $\mathbf{Set}^2$ . As suggested by Example 2.4.2., choose  $\mathbb{A} = \{(\Sigma, \emptyset) : \Sigma \in \mathbf{Set}_f\}$ . A finite  $\mathbf{T}_\infty$ -algebra (= finite  $\omega$ -semigroup)  $A$  is  $\mathbb{A}$ -generated iff it is *complete*, i.e. every element  $a \in A_\omega$  can be expressed as an infinite product  $a = \pi(a_0, a_1, \dots)$  for some  $a_i \in A_+$ . Clearly complete  $\omega$ -semigroups are closed under finite products. Thus a pseudovariety of  $\mathbf{T}_\infty$ -algebras is a class of finite complete  $\omega$ -semigroups closed under quotients, complete  $\omega$ -subsemigroups, and finite products. This is the concept studied by Wilke [36].



► **Remark 4.8.** Every  $\mathbf{T}$ -homomorphism  $g: \mathbf{T}D' \rightarrow \mathbf{T}D$  with  $D, D' \in \mathcal{D}_f^S$  extends uniquely to a  $\widehat{\mathbf{T}}$ -homomorphism  $\hat{g}: \widehat{\mathbf{T}}D' \rightarrow \widehat{\mathbf{T}}D$  with  $\iota_D \cdot g = V\hat{g} \cdot \iota_{D'}$ , cf. Remark 2.11.2.

► **Definition 4.9.** A *profinite theory* is a family  $\varphi = (\varphi_\Sigma: \widehat{\mathbf{T}}\Sigma \rightarrow P_\Sigma)_{\Sigma \in \mathbb{A}}$  of  $\Sigma$ -generated profinite  $\widehat{\mathbf{T}}$ -algebras such that for every  $\mathbf{T}$ -homomorphism  $g: \mathbf{T}\Delta \rightarrow \mathbf{T}\Sigma$  with  $\Sigma, \Delta \in \mathbb{A}$  there exists a  $\widehat{\mathbf{T}}$ -homomorphism  $g_P: P_\Delta \rightarrow P_\Sigma$  with  $\varphi_\Sigma \cdot \hat{g} = g_P \cdot \varphi_\Delta$ . Profinite theories are ordered by  $\varphi \leq \varphi'$  iff  $\varphi_\Sigma$  factors through  $\varphi'_\Sigma$  for each  $\Sigma \in \mathbb{A}$ .

► **Proposition 4.10.** *The poset of pseudovarieties of  $\mathbf{T}$ -algebras (ordered by inclusion) is isomorphic to the poset of profinite theories.*

**Proof sketch.** For any profinite theory  $\varphi = (\varphi_\Sigma: \widehat{\mathbf{T}}\Sigma \rightarrow P_\Sigma)_{\Sigma \in \mathbb{A}}$ , form the pseudovariety  $\mathcal{V}_\varphi$  of all finite  $\mathbf{T}$ -algebras  $(A, \alpha)$  whose corresponding  $\widehat{\mathbf{T}}$ -algebra  $(A, \alpha^+)$ , cf. Remark 2.11.1, is a quotient of some  $P_\Sigma$ . The map  $\varphi \mapsto \mathcal{V}_\varphi$  gives the isomorphism. ◀

► **Remark 4.11.** Again this result has an equational version. A finite  $\mathbf{T}$ -algebra  $A$  satisfies a profinite equation  $u = v$  over  $\Sigma \in \mathbb{A}$  if  $e^+(u) = e^+(v)$  for all surjective  $\mathbf{T}$ -homomorphisms  $e: \mathbf{T}\Sigma \rightarrow A$ . Pseudovarieties are the classes of  $\mathbb{A}$ -generated finite  $\mathbf{T}$ -algebras presentable by profinite equations over  $\mathbb{A}$ . For  $\mathbb{A} = \mathbf{Set}_f^S$ , this was proved in [13, Thm. 4.12 and Rem. 5.7].

## 5 The Variety Theorem

In this section we present our main result, the variety theorem for  $\mathbf{T}$ -recognizable languages.

► **Remark 5.1.** Recall that the variety  $\mathcal{C}$  is assumed to have a constant in the signature. Choosing a constant gives a natural transformation from  $C_1: \mathcal{C} \rightarrow \mathcal{C}$ , the constant functor on  $1 \in \mathcal{C}$ , to the identity functor  $\text{Id}_{\mathcal{C}}$ . It dualizes to a natural transformation  $\perp: \text{Id}_{\widehat{\mathcal{D}}} \rightarrow C_{O_{\mathcal{D}}}$ . The idea is that  $\perp$  models the empty set. For the categories  $\mathcal{D}$  of Example 2.2 and 2.3 we have  $O_{\mathbf{Set}} = \{0, 1\}$ ,  $O_{\mathbf{Pos}} = O_{\mathbf{JSL}_0} = \{0 < 1\}$  (the two-chain) and  $O_{\mathbf{Vec}_K} = K$ , and in each case we choose  $\perp: D \rightarrow O_{\mathcal{D}}$  for  $D \in \widehat{\mathcal{D}}$  to be the constant morphism with value 0.

► **Notation 5.2.** For each  $\Sigma \in \mathbf{Set}_f^S$  fix a unary presentation  $\mathbb{U}_\Sigma$  of  $\mathbf{T}$  over  $\Sigma$ .

► **Definition 5.3.** Let  $L: T\Sigma \rightarrow O_{\mathcal{D}}$  be a language over  $\Sigma \in \mathbf{Set}_f^S$ .

1. The *derivative*  $u^{-1}L$  of  $L$  w.r.t. an operation  $u: (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbb{U}_\Sigma$  is the language over  $\Sigma$  given on sort  $s$  by  $L_{s'} \cdot u: (T\Sigma)_s \rightarrow O_{\mathcal{D}}$  and on sorts  $t \neq s$  by  $V\perp \cdot \iota_{\Sigma}: (T\Sigma)_t \rightarrow O_{\mathcal{D}}$ .
2. The *preimage*  $g^{-1}L$  of  $L$  under a  $\mathbf{T}$ -homomorphism  $g: \mathbf{T}\Delta \rightarrow \mathbf{T}\Sigma$  is the language over  $\Delta$  defined by  $L \cdot g: T\Delta \rightarrow O_{\mathcal{D}}$ .

- **Example 5.4.** 1.  $\mathbf{T} = \mathbf{T}_*$  on  $\mathbf{Set}$ : let  $\mathbb{U}_\Sigma$  as in Example 3.11.1. The derivatives of  $L \subseteq \Sigma^*$  w.r.t. the operations in  $\mathbb{U}_\Sigma$  are the languages  $x^{-1}Ly^{-1} = \{v \in \Sigma^* : xvy \in L\}$ , where  $x, y \in \Sigma^*$ . These are the classical derivatives for languages of finite words.
2.  $\mathbf{T} = \mathbf{T}_\infty$  on  $\mathbf{Set}^2$ : let  $\mathbb{U}_{\overline{\Sigma}}$  (with  $\overline{\Sigma} = (\Sigma, \emptyset)$ ) as in Example 3.11.2. The derivatives of  $L \subseteq \Sigma^+ \cup \Sigma^\omega$  w.r.t. the operations in  $\mathbb{U}_{\overline{\Sigma}}$  are the languages  $\{v \in \Sigma^+ : xvy \in L\}$ ,  $\{v \in \Sigma^+ : xvx \in L\}$ ,  $\{v \in \Sigma^+ : x(vy)^\omega \in L\}$ , and  $\{v \in \Sigma^\omega : xv \in L\}$ , where  $x, y \in \Sigma^*$  and  $z \in \Sigma^\omega$ . These are the derivatives for  $\infty$ -languages studied by Wilke [36].
3. Let  $\mathbf{T}$  be a monad on  $\mathbf{Set}^S$ , and take the polynomial presentation  $\mathbb{U}_\Sigma$  of Example 3.11.3. The derivatives of a language  $L \subseteq T\Sigma$  w.r.t.  $\mathbb{U}_\Sigma$  are the languages  $p^{-1}L \subseteq T\Sigma$  with  $(p^{-1}L)_s = \{v \in (T\Sigma)_s : [p](v) \in L_{s'}\}$  and  $(p^{-1}L)_t = \emptyset$  for  $t \neq s$ , where  $p: 1_{s'} \rightarrow T(\Sigma + 1_s)$  is a polynomial over  $\Sigma$ . These are the *polynomial derivatives* studied by Bojańczyk [12].

► **Proposition 5.5.** *Derivatives and preimages of recognizable languages are recognizable.*

► **Remark 5.6.** Recall the isomorphism  $\text{Rec}(\Sigma) \cong \prod_s P(\hat{T}\Sigma)_s$  of Remark 3.4. In the following we study subobjects  $V_\Sigma \subseteq \text{Rec}(\Sigma)$  in  $\mathcal{C}$ . For technical reasons we restrict ourselves to subobjects of the form  $\prod_s m_s: \prod_s (V'_\Sigma)_s \rightarrow \prod_s P(\hat{T}\Sigma)_s$ , where  $m_s: (V'_\Sigma)_s \rightarrow P(\hat{T}\Sigma)_s$  lives in  $\mathcal{C}$ . Such subobjects are called *admissible*. Clearly, for  $S = 1$ , any subobject of  $\text{Rec}(\Sigma)$  is admissible.

$$\begin{array}{ccc} V_\Sigma & \xrightarrow{\subseteq} & \text{Rec}(\Sigma) \\ \cong \downarrow & & \downarrow \cong \\ \prod_s (V'_\Sigma)_s & \xrightarrow{\prod_s m_s} & \prod_s P(\hat{T}\Sigma)_s \end{array}$$

► **Definition 5.7.** 1. A *local variety of languages over  $\Sigma \in \mathbf{Set}_f^S$*  is an admissible subobject  $V_\Sigma \subseteq \text{Rec}(\Sigma)$  closed under derivatives, i.e.  $L \in V_\Sigma$  implies  $u^{-1}L \in V_\Sigma$  for all  $u \in \mathbf{U}_\Sigma$ .  
2. A *variety of languages* is a family  $(V_\Sigma \subseteq \text{Rec}(\Sigma))_{\Sigma \in \mathbb{A}}$  of local varieties closed under preimages, i.e.  $L \in V_\Sigma$  implies  $g^{-1}L \in V_\Delta$  for all  $\Sigma, \Delta \in \mathbb{A}$  and  $g: \mathbf{T}\Delta \rightarrow \mathbf{T}\Sigma$  in  $\mathbf{Alg} \mathbf{T}$ .

► **Remark 5.8.** As mentioned above, the condition in Definition 5.7.1 that  $V_\Sigma$  is admissible is trivial for  $S = 1$ . More importantly, if  $\mathbf{U}_\Sigma$  contains all identity morphisms, this condition can also be dropped in the many-sorted case for the categories  $\mathcal{C}$  of Example 2.2 and 2.3, since one can show that any subobject  $V_\Sigma \subseteq \text{Rec}(\Sigma)$  closed under derivatives is admissible.

We are ready to state the main result of our paper, which holds under the Assumptions 2.1.

► **Theorem 5.9 (Variety Theorem).** 1. *The poset of varieties of languages (ordered by inclusion) is isomorphic to the poset of pseudovarieties of  $\mathbf{T}$ -algebras.*  
2. *For each  $\Sigma \in \mathbf{Set}_f^S$ , the poset of local varieties of languages over  $\Sigma$  (ordered by inclusion) is isomorphic to the poset of local pseudovarieties of  $\Sigma$ -generated  $\mathbf{T}$ -algebras.*

**Proof sketch.** Duality! For the second isomorphism one shows that an admissible subobject  $V_\Sigma \subseteq \text{Rec}(\Sigma)$ , represented by a morphism  $(m_s: (V'_\Sigma)_s \rightarrow P(\hat{T}\Sigma)_s)_{s \in S}$  in  $\mathcal{C}^S$ , is closed under derivatives iff its dual morphism  $(P^{-1}m_s: (\hat{T}\Sigma)_s \rightarrow P^{-1}(V'_\Sigma)_s)_{s \in S}$  in  $\hat{\mathcal{D}}^S$  carries a  $\Sigma$ -generated profinite  $\hat{\mathbf{T}}$ -algebra. Then Proposition 4.3 gives the isomorphism. For the first isomorphism, one shows that a family  $(V_\Sigma)_{\Sigma \in \mathbb{A}}$  of local varieties is closed under preimages iff its dual family forms a profinite theory. Then Proposition 4.10 gives the isomorphism. ◀

► **Remark 5.10.** Straubing [35] studied *C-varieties of regular languages* which are defined as Eilenberg's varieties of regular languages, except that closure under preimages is required only w.r.t. a given class  $\mathbf{C}$  of monoid morphisms. By making a class  $\mathbf{C}$  of  $\mathbf{T}$ -homomorphisms an additional parameter of our framework, Theorem 5.9 (and its duality-based proof) easily generalize to a monad version of Straubing's variety theorem for  $\mathbf{C}$ -varieties.

## 6 Applications

We now apply Theorem 5.9 to various monads to derive concrete Eilenberg correspondences.

(a) **Languages of finite words.** Let  $\mathcal{D}$  be a *commutative* variety of algebras or ordered algebras, i.e. for any two objects  $A, B \in \mathcal{D}$  the hom-set  $\mathcal{D}(A, B)$  carries a subobject of  $B^{|A|}$  in  $\mathcal{D}$ . All varieties  $\mathcal{D}$  of Example 2.2 and 2.3 are commutative. A  $\mathcal{D}$ -monoid is an object  $D \in \mathcal{D}$  with a monoid structure  $(|D|, \bullet, 1)$  on the underlying set such that the multiplication is a *bimorphism*; that is, for every  $x \in |D|$  the maps  $x \bullet -: |D| \rightarrow |D|$  and  $- \bullet x: |D| \rightarrow |D|$  carry endomorphisms of  $D$ . Let  $\mathbf{T}_M$  be the monad on  $\mathcal{D}$  constructing free  $\mathcal{D}$ -monoids. In [1] we showed that the free  $\mathcal{D}$ -monoid on  $\Sigma \in \mathcal{D}$  is  $(\Sigma^*, \bullet, \varepsilon)$ , where  $\Sigma^*$  is the free  $\mathcal{D}$ -object on the set  $\Sigma^*$ ,  $\bullet$  extends the concatenation of words, and  $\varepsilon$  is the empty word. Thus  $T_M \Sigma = \Sigma^*$ . A language  $L: T_M \Sigma \rightarrow \mathcal{O}_\mathcal{D}$  is  $\mathbf{T}_M$ -recognizable iff its adjoint transpose  $L': \Sigma^* \rightarrow |\mathcal{O}_\mathcal{D}|$  (via the right adjoint  $|-|: \mathcal{D} \rightarrow \mathbf{Set}$ ) is regular, i.e. computed by some finite Moore automaton

with output set  $|O_{\mathcal{D}}|$ . Generalizing Example 3.6.1, we showed in [2] that each recognizable language  $L: \Sigma^* \rightarrow O_{\mathcal{D}}$  has a syntactic  $\mathcal{D}$ -monoid  $e_L: \Sigma^* \rightarrow \Sigma^*/\equiv_L$ , where  $v \equiv_L w$  iff  $L(x \bullet v \bullet y) = L(x \bullet w \bullet y)$  for all  $x, y \in \Sigma^*$ . For ordered varieties  $\mathcal{D}$ , e.g. **Pos**, one uses in lieu of  $\equiv_L$  the preorder  $\leq_L$  on  $\Sigma^*$  defined by  $v \leq_L w$  iff  $L(x \bullet v \bullet y) \leq L(x \bullet w \bullet y)$  for all  $x, y \in \Sigma^*$ , and forms the induced poset  $\Sigma^*/\leq_L$ . Theorem 3.9 and Remark 3.12 give the unary presentation  $\mathbb{U}_{\Sigma} = \{ \Sigma^* \xrightarrow{x \bullet \bullet y} \Sigma^* : x, y \in \Sigma^* \}$  for all  $\Sigma \in \mathbf{Set}_f$ . A *variety of regular languages* in  $\mathcal{C}$  associates to each  $\Sigma$  a set of regular languages over  $\Sigma$  closed under  $\mathcal{C}$ -algebraic operations (see Remark 3.4), derivatives (see Example 5.4.1) and preimages of  $\mathcal{D}$ -monoid morphisms. Theorem 5.9 then instantiates to the main results of our papers [1, 3, 14]:

► **Theorem 6.1.** *The poset of (local) varieties of regular languages in  $\mathcal{C}$  is isomorphic to the poset of (local) pseudovarieties of  $\mathcal{D}$ -monoids.*

For the categories of Example 2.2 and 2.3 we get the Eilenberg theorems listed below. The third column describes the  $\mathcal{C}$ -algebraic operations under which (local) varieties of languages are closed, and the fourth column gives the type of  $\mathcal{D}$ -monoids under consideration. All these correspondences are known in the literature, and are uniformly covered by Theorem 6.1.

| $\mathcal{C}$          | $\mathcal{D}$          | (local) var. of lang. closed under | $\cong$ | (local) pseudovarieties of | proved in |
|------------------------|------------------------|------------------------------------|---------|----------------------------|-----------|
| <b>BA</b>              | <b>Set</b>             | boolean operations                 |         | monoids                    | [16, 18]  |
| <b>DL<sub>01</sub></b> | <b>Pos</b>             | union and intersection             |         | ordered monoids            | [18, 24]  |
| <b>JSL<sub>0</sub></b> | <b>JSL<sub>0</sub></b> | union                              |         | idempotent semirings       | [28]      |
| <b>Vec<sub>K</sub></b> | <b>Vec<sub>K</sub></b> | addition of weighted languages     |         | $K$ -algebras              | [31]      |

(b) **Polynomial varieties.** Let  $\mathbf{T}$  be any monad on  $\mathbf{Set}^S$ . Choose  $\mathbb{A} = \mathbf{Set}_f^S$  and  $\mathbb{U}_{\Sigma}$  as in Example 3.11.3. A *polynomial variety of  $\mathbf{T}$ -recognizable languages* associates to each  $\Sigma \in \mathbf{Set}_f^S$  a set of  $\mathbf{T}$ -recognizable languages over  $\Sigma$  closed under boolean operations, polynomial derivatives (see Example 5.4.3), and preimages of  $\mathbf{T}$ -homomorphisms. Theorem 5.9 yields the following Eilenberg correspondence. Its non-local part is due to Bojańczyk [12].

► **Theorem 6.2.** *The poset of (local) polynomial varieties of  $\mathbf{T}$ -recognizable languages is isomorphic to the poset of (local) pseudovarieties of  $\mathbf{T}$ -algebras.*

Next, we consider correspondences that are *not* covered by Theorem 6.1 and 6.2, but are either instances of Theorem 5.9, or emerge by introducing new parameters to our setting.

(c) **Languages of  $\infty$ -words.** Let  $\mathbf{T} = \mathbf{T}_{\infty}$  on  $\mathbf{Set}^2$  with  $\mathbb{A} = \{ (\Sigma, \emptyset) : \Sigma \in \mathbf{Set}_f \}$ , and consider the unary presentation of Example 3.11.2. A *variety of  $\infty$ -languages* associates to each  $\Sigma \in \mathbf{Set}_f$  a set of regular  $\infty$ -languages over  $\Sigma$  closed under boolean operations, derivatives (see Example 5.4.2) and preimages of  $\omega$ -semigroup morphisms. Theorem 5.9 gives

► **Theorem 6.3.** *The poset of (local) varieties of  $\infty$ -languages is isomorphic to the poset of (local) pseudovarieties of  $\omega$ -semigroups.*

The non-local part is Wilke's theorem for  $\infty$ -languages [36] (in the formulation of [23]), while the local part is a new result, extending the corresponding result of Gehrke, Grigorieff, and Pin [18] for finite words. Similarly, one can take the monad  $\mathbf{T}_{\infty, \leq}$  on **Pos** representing *ordered  $\omega$ -semigroups*. Since  $\mathcal{C} = \mathbf{DL}_{01}$ , we obtain *positive varieties of  $\infty$ -languages*, emerging from Wilke's concept by dropping closure under complement. Then Theorem 5.9 gives the result below. Its non-local part is due to Pin [25], and the local part is again a new result.

► **Theorem 6.4.** *The poset of (local) positive varieties of  $\infty$ -languages is isomorphic to the poset of (local) pseudovarieties of ordered  $\omega$ -semigroups.*

The next three examples can be treated using the same techniques as above; we postpone the details to a full journal version of this paper.

**(d) Ordered words.** A natural generalization of  $\infty$ -words are words on linear orderings, for which Bedon et al. [8,9] establish two variety theorems. Both follow from Theorem 5.9.

**(e) Tree languages.** Languages of binary trees are represented by the monad  $\mathbf{T}$  on  $\mathbf{Set}^3$  associated to Wilke’s *tree algebras* [37]. The free tree algebra on  $(\Sigma, \emptyset, \emptyset)$  is  $T(\Sigma, \emptyset, \emptyset) = (\Sigma, T_\Sigma, C_\Sigma)$  where  $T_\Sigma$  is the set of  $\Sigma$ -labeled finite binary trees (labeled at every node) and  $C_\Sigma$  is the set of *contexts*, i.e.  $(\Sigma + \{*\})$ -labeled binary trees where  $*$  appears only at a single leaf. We take  $\mathbb{A} = \{(\Sigma, \emptyset, \emptyset) : \Sigma \in \mathbf{Set}_f\}$ . Tree languages are subsets of  $T_\Sigma$ , or equivalently, subsets of  $T(\Sigma, \emptyset, \emptyset)$  that are empty in the first and third sort. On the algebraic side, one needs to restrict to *reduced* tree algebras. These are  $\mathbb{A}$ -generated  $\mathbf{T}$ -algebras  $A$  determined by the second sort, in the sense that a quotient  $e: A \rightarrow B$  is an isomorphism whenever it is an isomorphism in the second sort. Salehi and Steinby [33] prove a correspondence between varieties of tree languages and pseudovarieties of reduced tree algebras. This is not a direct instance of Theorem 5.9, as languages are restricted to a subset of the sorts. However, by making our setting parametric in a subset  $S_0 \subseteq S$ , we can cover this result with our methods.

**(f) Cost functions.** Daviaud, Kuperberg, and Pin [15] study varieties of *regular cost functions*, a quantitative version of regular languages. The corresponding algebras are called *stabilization algebras*. These are ordered algebras whose axioms involve inequations but also an implication. Consequently stabilization algebras do not form a variety of ordered algebras and are not represented by a monad on  $\mathbf{Pos}$ . However, one can take the monad  $\mathbf{T}_S$  on  $\mathbf{Pos}$  associated to the theory of stabilization algebras *minus* the implication. Then, as shown in [15], regular cost functions correspond to languages  $L: T_S \Sigma \rightarrow \{0 < 1\}$  recognized by finite stabilization algebras (rather than *arbitrary* finite  $\mathbf{T}_S$ -algebras).

To cover stabilization algebras in our categorical setting, we need an additional parameter: a *quasivariety*  $\mathcal{Q} \subseteq \mathbf{Alg}_f \mathbf{T}$  of finite  $\mathbf{T}$ -algebras, i.e. a subclass closed under subalgebras and finite products. (In the above example,  $\mathcal{Q}$  is taken to be the quasivariety of all finite stabilization algebras, that is, finite  $\mathbf{T}_S$ -algebras satisfying the implication.) In lieu of the profinite monad  $\hat{\mathbf{T}}$  we form the *pro- $\mathcal{Q}$  monad*  $\hat{\mathbf{T}}_{\mathcal{Q}}$  on  $\hat{\mathcal{D}}^S$ , where  $\hat{\mathbf{T}}_{\mathcal{Q}} D$  for  $D \in \mathcal{D}_f^S$  is the inverse limit of all quotients of  $\mathbf{T}D$  in  $\mathcal{Q}$ . Profinite  $\hat{\mathbf{T}}$ -algebras are replaced by *pro- $\mathcal{Q}$  algebras* for  $\hat{\mathbf{T}}_{\mathcal{Q}}$ , i.e. quotient algebras of  $\hat{\mathbf{T}}_{\mathcal{Q}}$  arising as inverse limits of algebras in  $\mathcal{Q}$ . A *pseudovariety of  $\mathbf{T}$ -algebras relative to  $\mathcal{Q}$*  is a subclass of  $\mathcal{Q}$  closed under quotients (in  $\mathcal{Q}$ ) and  $\mathbb{A}$ -generated subalgebras of finite products. Theorem 5.9 easily generalizes to a correspondence between varieties of  $\mathcal{Q}$ -recognizable languages and pseudovarieties of  $\mathbf{T}$ -algebras relative to  $\mathcal{Q}$ . For the monad  $\mathbf{T}_S$  on  $\mathbf{Pos}$  and  $\mathcal{Q} =$  finite stabilization algebras, we recover the variety theorem of [15]: varieties of cost functions correspond to pseudovarieties of stabilization algebras.

## 7 Conclusions and Future Work

We presented a duality-based framework for algebraic language theory that captures, to the best of our knowledge, the bulk of Eilenberg theorems in the literature. Besides working out the details of (d)–(f) above, there are a few interesting directions for future work. First, we aim to apply our framework to *nominal* Stone duality [17], possibly leading to a variety theory for data languages. Second, we aim to investigate additional parameters, e.g. use an abstract factorization system in  $\mathcal{D}$ , and use in lieu of free objects  $\Sigma$  arbitrary finite objects  $X \in \mathcal{D}_f^S$  as “alphabets”. This allows e.g. to study the free-category monad on the category of graphs: we expect to obtain a variety theorem for languages of finite paths vs. pseudovarieties of categories, a counterpart to the Reiterman theorem for finite categories of Jones [20].

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This appendix contains all proofs and additional details we omitted due to space restrictions.

## A Categorical toolkit

We review some concepts from category theory we will use throughout this paper. For details we refer to standard textbooks such as [22], and also to [4] for an introduction to locally finitely copresentable categories.

► **A.1. Monads.** A *monad* on a category  $\mathcal{A}$  is a triple  $\mathbf{T} = (T, \eta, \mu)$  consisting of an endofunctor  $T: \mathcal{A} \rightarrow \mathcal{A}$  and two natural transformations  $\eta: \text{Id} \rightarrow T$  and  $\mu: TT \rightarrow T$  (called the *unit* and *multiplication* of  $\mathbf{T}$ ) such that the following diagrams commute:

$$\begin{array}{ccc} T & \xrightarrow{\eta T} & TT \\ & \searrow id & \downarrow \mu \\ & & T \end{array} \quad \begin{array}{ccc} T & \xrightarrow{T\eta} & TT \\ & \searrow id & \downarrow \mu \\ & & T \end{array} \quad \begin{array}{ccc} TTT & \xrightarrow{T\mu} & TT \\ \mu T \downarrow & & \downarrow \mu \\ TT & \xrightarrow{\mu} & T \end{array}$$

Given two monads  $\mathbf{S} = (S, \eta^{\mathbf{S}}, \mu^{\mathbf{S}})$  and  $\mathbf{T} = (T, \eta^{\mathbf{T}}, \mu^{\mathbf{T}})$  on  $\mathcal{A}$ , a *monad morphism*  $\varphi: \mathbf{S} \rightarrow \mathbf{T}$  is a natural transformation  $\varphi: S \rightarrow T$  making the following diagrams commute:

$$\begin{array}{ccc} \text{Id} & \xrightarrow{\eta^{\mathbf{S}}} & S \\ & \searrow \eta^{\mathbf{T}} & \downarrow \varphi \\ & & T \end{array} \quad \begin{array}{ccccc} SS & \xrightarrow{S\varphi} & ST & \xrightarrow{\varphi T} & TT \\ \mu^{\mathbf{S}} \downarrow & & \downarrow \mu^{\mathbf{S}} & & \downarrow \mu^{\mathbf{T}} \\ S & \xrightarrow{\varphi} & T & & T \end{array}$$

► **A.2. Algebras for a monad.** Let  $\mathbf{T} = (T, \eta, \mu)$  be a monad on a category  $\mathcal{A}$ . By a  $\mathbf{T}$ -*algebra* is meant a pair  $(A, \alpha)$  of an object  $A \in \mathcal{A}$  and a morphism  $\alpha: TA \rightarrow A$  satisfying the *unit* and *associative law*:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow id & \downarrow \alpha \\ & & A \end{array} \quad \begin{array}{ccc} TTA & \xrightarrow{\mu_A} & TA \\ T\alpha \downarrow & & \downarrow \alpha \\ TA & \xrightarrow{\alpha} & A \end{array}$$

Given two  $\mathbf{T}$ -algebras  $(A, \alpha)$  and  $(B, \beta)$ , a  $\mathbf{T}$ -*homomorphism*  $h: (A, \alpha) \rightarrow (B, \beta)$  is a morphism  $h: A \rightarrow B$  in  $\mathcal{A}$  such that  $h \cdot \alpha = \beta \cdot Th$ . Denote by  $\mathbf{Alg} \mathbf{T}$  the category of  $\mathbf{T}$ -algebras and  $\mathbf{T}$ -homomorphisms. There is a forgetful functor  $U: \mathbf{Alg} \mathbf{T} \rightarrow \mathcal{A}$  given by  $(A, \alpha) \mapsto A$  on objects and  $h \mapsto h$  on morphisms. It has a left adjoint assigning to each object  $A$  of  $\mathcal{A}$  the  $\mathbf{T}$ -algebra  $\mathbf{T}A = (TA, \mu_A)$ , called the *free  $\mathbf{T}$ -algebra on  $A$* , and to each morphism  $h: A \rightarrow B$  the  $\mathbf{T}$ -homomorphism  $Th: \mathbf{T}A \rightarrow \mathbf{T}B$ . Note that for any  $\mathbf{T}$ -algebra  $(A, \alpha)$  the associative law states precisely that  $\alpha: \mathbf{T}A \rightarrow (A, \alpha)$  is a  $\mathbf{T}$ -homomorphism.

► **A.3. Limits of  $\mathbf{T}$ -algebras.** The forgetful functor  $U: \mathbf{Alg} \mathbf{T} \rightarrow \mathcal{A}$  preserves limits, being a right adjoint (see A.2). More importantly, it also *creates limits*. That is, given a diagram  $D: \mathcal{S} \rightarrow \mathbf{Alg} \mathbf{T}$  and a limit cone  $(p_s: A \rightarrow UD_s)_{s \in \mathcal{S}}$  over  $UD$  in  $\mathcal{A}$ , there exists a unique  $\mathbf{T}$ -algebra structure  $(A, \alpha)$  on  $A$  such that all  $p_s$  are  $\mathbf{T}$ -homomorphisms, and moreover  $(p_s: (A, \alpha) \rightarrow D_s)_{s \in \mathcal{S}}$  forms a limit cone over  $D$  in  $\mathbf{Alg} \mathbf{T}$ . In case  $\mathcal{A}$  is complete (that is, it has all limits), it follows that  $\mathbf{Alg} \mathbf{T}$  is complete and that  $U$  *reflects limits*. That is, a cone  $(p_s: (A, \alpha) \rightarrow D_s)_{s \in \mathcal{S}}$  over  $D$  is a limit cone if  $(p_s: A \rightarrow UD_s)_{s \in \mathcal{S}}$  is a limit cone over  $UD$ .

► **A.4. Comma categories.** Let  $F: \mathcal{B} \rightarrow \mathcal{A}$  be a functor and  $A$  an object in  $\mathcal{A}$ . The *comma category*  $(A \downarrow F)$  has as objects all morphisms  $(A \xrightarrow{f} FB)$  in  $\mathcal{A}$  with  $B \in \mathcal{B}$ , and its morphisms from  $(A \xrightarrow{f_1} FB_1)$  to  $(A \xrightarrow{f_2} FB_2)$  are morphisms  $h: B_1 \rightarrow B_2$  in  $\mathcal{B}$  with  $f_2 = Fh \cdot f_1$ . If  $F: \mathcal{B} \hookrightarrow \mathcal{A}$  is the inclusion of a subcategory  $\mathcal{B}$ , we write  $(A \downarrow \mathcal{B})$  for  $(A \downarrow F)$ .

► **A.5. Kan extensions.** The *right Kan extension* of a functor  $F: \mathcal{A} \rightarrow \mathcal{C}$  along  $K: \mathcal{A} \rightarrow \mathcal{B}$  is a functor  $R: \mathcal{B} \rightarrow \mathcal{C}$  together with a universal natural transformation  $\varepsilon: RK \rightarrow F$ , i.e. for every functor  $G: \mathcal{B} \rightarrow \mathcal{C}$  and every natural transformation  $\gamma: GK \rightarrow F$  there exists a unique natural transformation  $\gamma^\dagger: G \rightarrow R$  with  $\gamma = \varepsilon \cdot \gamma^\dagger K$ . If  $\mathcal{A}$  is small and  $\mathcal{C}$  is complete, the object  $RB$  for  $B \in \mathcal{B}$  is the limit of the diagram

$$(B \downarrow K) \xrightarrow{Q^B} \mathcal{A} \xrightarrow{F} \mathcal{C}$$

that maps  $(B \xrightarrow{f} KA)$  to  $FA$  and  $h: (B \xrightarrow{f_1} KA_1) \rightarrow (B \xrightarrow{f_2} KA_2)$  to  $Fh$ .

► **A.6. Codensity monads.** Let  $\varepsilon: RK \rightarrow K$  be the right Kan extension of a functor  $K: \mathcal{A} \rightarrow \mathcal{B}$  along itself. Then  $R$  can be equipped with a monad structure  $\mathbf{R} = (R, \eta^{\mathbf{R}}, \mu^{\mathbf{R}})$  where the unit  $\eta^{\mathbf{R}}$  is  $(id_K)^\dagger: \text{Id} \rightarrow R$  and the multiplication  $\mu^{\mathbf{R}}$  is  $(\varepsilon \cdot R\varepsilon)^\dagger: RR \rightarrow R$ . The monad  $\mathbf{R}$  is called the *codensity monad* of  $K$ , see e.g. [21].

► **A.7. Cofiltered limits and inverse limits.** A category  $\mathcal{K}$  is *cofiltered* if every finite subcategory of  $\mathcal{K}$  has a cone in  $\mathcal{K}$ . This is equivalent to the following three conditions:

- (i)  $\mathcal{K}$  is nonempty.
- (ii) For any two objects  $Y$  and  $Z$  of  $\mathcal{K}$ , there exist two morphisms  $f: X \rightarrow Y$  and  $g: X \rightarrow Z$  with a common domain  $X$ .
- (iii) For any two parallel morphisms  $f, g: Y \rightarrow Z$  in  $\mathcal{K}$ , there exists a morphism  $e: X \rightarrow Y$  with  $f \cdot e = g \cdot e$ .

A *cofiltered limit* in a category  $\mathcal{A}$  is a limit of a diagram  $\mathcal{K} \rightarrow \mathcal{A}$  with cofiltered scheme  $\mathcal{K}$ . It is also called an *inverse limit* if  $\mathcal{K}$  is a (co-directed) poset.

The dual concept of a cofiltered limit is a *filtered colimit*.

► **A.8. Final functors.** A functor  $F: \mathcal{K} \rightarrow \mathcal{B}$ , where  $\mathcal{K}$  is cofiltered, is called *final* if

- (i) for any object  $B$  of  $\mathcal{B}$ , there exists a morphism  $f: FK \rightarrow B$  for some  $K \in \mathcal{K}$ , and
- (ii) given two parallel morphisms  $f, g: FK \rightarrow B$  with  $K \in \mathcal{K}$ , there exists a morphism  $k: K' \rightarrow K$  in  $\mathcal{K}$  with  $f \cdot Fk = g \cdot Fk$ .

The importance of final functors is that they facilitate the construction of limits. If  $F: \mathcal{K} \rightarrow \mathcal{B}$  is final, a diagram  $D: \mathcal{B} \rightarrow \mathcal{A}$  has a limit iff the diagram  $DF: \mathcal{K} \rightarrow \mathcal{A}$  has a limit, and in this case the two limits agree. Specifically, any limit cone  $(p_B: A \rightarrow D_B)_{B \in \mathcal{B}}$  over  $D$  restricts to a limit cone  $(p_{FK}: A \rightarrow D_{FK})_{K \in \mathcal{K}}$  over  $DF$ .

► **A.9. Finitely copresentable objects.** An object  $A$  of a category  $\mathcal{A}$  is called *finitely copresentable* if the hom-functor  $\mathcal{A}(-, A): \mathcal{A}^{op} \rightarrow \mathbf{Set}$  preserves filtered colimits. Equivalently, for any cofiltered limit cone  $(p_i: B \rightarrow B_i)_{i \in I}$  in  $\mathcal{A}$  the following two statements hold:

- (i) Every morphism  $f: B \rightarrow A$  factors through some  $p_i$ .
- (ii) Given  $i \in I$  and morphisms  $s, s': B_i \rightarrow A$  with  $s \cdot p_i = s' \cdot p_i$ , there exists a morphism  $b_{ji}: B_j \rightarrow B_i$  in the diagram with  $s \cdot b_{ji} = s' \cdot b_{ji}$ .

► **A.10. Locally finitely copresentable categories.** A category  $\mathcal{A}$  is called *locally finitely copresentable* if it satisfies the following three properties:

- (i)  $\mathcal{A}$  is complete;

- (ii) the full subcategory  $\mathcal{A}_f$  of finitely copresentable objects is essentially small, i.e. the objects of  $\mathcal{A}_f$  (taken up to isomorphism) form a set;
- (iii) any object  $A$  of  $\mathcal{A}$  is a cofiltered limit of finitely copresentable objects; that is, there exists a cofiltered limit cone  $(A \rightarrow A_i)_{i \in I}$  in  $\mathcal{A}$  with  $A_i \in \mathcal{A}_f$  for all  $i \in I$ .

If  $\mathcal{A}$  is locally finitely copresentable, so is any functor category  $\mathcal{A}^{\mathcal{S}}$ , where  $\mathcal{S}$  is an arbitrary small category. In particular, this holds for any product category  $\mathcal{A}^S$  (where  $S$  is set) and for the arrow category  $\mathcal{A}^{\rightarrow}$ . The latter has as objects all morphisms of  $\mathcal{A}$ , and as morphism from  $(A \xrightarrow{f} B)$  to  $(C \xrightarrow{g} D)$  all pairs of morphisms  $(a : A \rightarrow C, b : B \rightarrow D)$  in  $\mathcal{A}$  with  $b \cdot f = g \cdot a$ . The finitely copresentable objects of  $\mathcal{A}^{\rightarrow}$  are precisely the arrows with finitely copresentable domain and codomain.

► **A.11. Cofiltered limits in locally finitely copresentable categories.** A cofiltered cone  $(p_i : B \rightarrow B_i)_{i \in I}$  in a locally finitely copresentable category  $\mathcal{A}$  is a limit cone iff

- (i) every morphism  $f : B \rightarrow A$  with  $A \in \mathcal{A}_f$  factors through some  $p_i$ , and
- (ii) this factorization is essentially unique: given  $i \in I$  and  $s, s' : B_i \rightarrow A$  with  $s \cdot p_i = s' \cdot p_i$ , there exists a morphism  $b_{ji} : B_j \rightarrow B_i$  in the diagram with  $s \cdot b_{ji} = s' \cdot b_{ji}$ .

Note that if all  $p_i$ 's are epimorphisms, condition (ii) is superfluous.

► **A.12. Canonical diagrams.** Let  $\mathcal{A}$  be a locally finitely copresentable category. Then for each object  $A \in \mathcal{A}$  the comma category  $(A \downarrow \mathcal{A}_f)$  is essentially small and cofiltered. The *canonical diagram* of  $A$  is the cofiltered diagram  $K_A : (A \downarrow \mathcal{A}_f) \rightarrow \mathcal{A}$  that maps an object  $(A \xrightarrow{f} A_1)$  to  $A_1$  and a morphism  $h : (A \xrightarrow{f_1} A_1) \rightarrow (A \xrightarrow{f_2} A_2)$  to  $h : A_1 \rightarrow A_2$ . Every object  $A$  of  $\mathcal{A}$  is the cofiltered limit of its canonical diagram, that is,  $K_A$  has the limit cone

$$(f : A \rightarrow K_A f)_{f \in (A \downarrow \mathcal{A}_f)}.$$

► **A.13. Pro-completions.** Let  $\mathcal{B}$  be a small category. By a *pro-completion* (or a *free completion under cofiltered limits*) of  $\mathcal{B}$  is meant a category  $\mathbf{Pro} \mathcal{B}$  together with a full embedding  $I : \mathcal{B} \hookrightarrow \mathbf{Pro} \mathcal{B}$  such that

- (i)  $\mathbf{Pro} \mathcal{B}$  has cofiltered limits.
- (ii) For any functor  $F : \mathcal{B} \rightarrow \mathcal{C}$  into a category  $\mathcal{C}$  with cofiltered limits, there exists a functor  $\bar{F} : \mathbf{Pro} \mathcal{B} \rightarrow \mathcal{C}$ , unique up to natural isomorphism, such that  $\bar{F}$  preserves cofiltered limits and  $\bar{F} \cdot I$  is naturally isomorphic to  $F$ .

The universal property determines  $\mathbf{Pro} \mathcal{B}$  uniquely up to equivalence of categories. If the category  $\mathcal{B}$  has finite limits, then  $\mathbf{Pro} \mathcal{B}$  is locally finitely copresentable, and its finitely copresentable objects are up to isomorphism the objects  $IB$  ( $B \in \mathcal{B}$ ). Conversely, every locally finitely copresentable category  $\mathcal{A}$  arises in this way: we have  $\mathcal{A} = \mathbf{Pro} \mathcal{A}_f$ .

The dual concept of a pro-completion is an *ind-completion*, the free completion under cofiltered limits.

► **A.14. Factorization systems.** A *factorization system* in a category  $\mathcal{A}$  is a pair  $(\mathcal{E}, \mathcal{M})$  where  $\mathcal{E}$  and  $\mathcal{M}$  are classes of morphisms of  $\mathcal{A}$  with the following properties:

- (i) Both  $\mathcal{E}$  and  $\mathcal{M}$  are closed under composition and contain all isomorphisms.
- (ii) Every morphism  $f$  of  $\mathcal{A}$  has a factorization  $f = m \cdot e$  with  $e \in \mathcal{E}$  and  $m \in \mathcal{M}$ .
- (iii) The *diagonal fill-in* property holds: given a commutative square as shown below with  $e \in \mathcal{E}$  and  $m \in \mathcal{M}$ , there exists a unique morphism  $d$  making both triangles commute.

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ f \downarrow & \swarrow d & \searrow g \\ C & \xrightarrow{m} & D \end{array}$$

We will use three standard facts about factorization systems:

- (a) Suppose that  $\mathcal{M}$  is a class of monomorphisms. If  $(p_i : A \rightarrow A_i)_{i \in I}$  is a limit cone in  $\mathcal{A}$ , then the factorization  $p_i = (A \xrightarrow{e_i} A'_i \xrightarrow{m_i} A_i)$  with  $e_i \in \mathcal{E}$  and  $m_i \in \mathcal{M}$  yields another limit cone  $(e_i : A \rightarrow A'_i)_{i \in I}$  over the same scheme.
- (b) Suppose that  $\mathcal{E}$  is a class of epimorphisms. If  $\mathbf{T}$  is a monad on  $\mathcal{A}$  that preserves  $\mathcal{E}$ , i.e.  $e \in \mathcal{E}$  implies  $Te \in \mathcal{E}$ , then  $\mathbf{Alg} \mathbf{T}$  has the factorization system of  $\mathcal{E}$ -carried and  $\mathcal{M}$ -carried  $\mathbf{T}$ -homomorphisms.
- (c) Every locally finitely copresentable category  $\mathcal{A}$  has the (epi, strong mono) factorization system. Its arrow category  $\mathcal{A}^\rightarrow$ , see A.10, has the factorization system of componentwise epimorphic and strongly monomorphic morphisms.

## B Topological toolkit

The following two lemmas give important properties of cofiltered limits in the category of compact Hausdorff spaces and continuous maps. For proofs see Chapter 1 of [32].

► **Lemma B.1.** *Let  $\tau : D_1 \rightarrow D_2$  be a natural transformation between cofiltered diagrams (of the same scheme) in the category of compact Hausdorff spaces. If each  $\tau_i : D_{1i} \rightarrow D_{2i}$  is surjective, so is the mediating map  $\text{Lim } \tau : \text{Lim } D_1 \rightarrow \text{Lim } D_2$ . In particular, if  $\tau_i : X \rightarrow D_i$  is a cone of surjections, then the mediating map  $h = \text{lim } \tau$  is surjective.*

► **Lemma B.2.** *Let  $D$  be a cofiltered diagram in the category of compact Hausdorff spaces. If all  $D(i \xrightarrow{f} j)$  are surjective, so is each limit projection  $\varrho_i : \text{Lim } D \rightarrow D_i$ .*

## C Details for Section 2

**Details for Remark 2.12.1.** The forgetful functor  $|-| : \widehat{\mathcal{D}} \rightarrow \mathbf{Set}$  is representable by  $\mathbf{1}$ , i.e. it is naturally isomorphic to  $\widehat{\mathcal{D}}(\mathbf{1}, -)$  via the isomorphisms

$$|D| \cong \mathbf{Set}(\{*\}, |D|) \cong \widehat{\mathcal{D}}(\mathbf{1}, D).$$

natural in  $D \in \widehat{\mathcal{D}}$ . Then, the natural isomorphism  $|P| \cong \widehat{\mathcal{D}}(-, O_{\mathcal{D}})$  follows from the observation that the diagram below commutes for all  $h : D' \rightarrow D$  in  $\widehat{\mathcal{D}}$ :

$$\begin{array}{ccc} \mathcal{C}(\mathbf{1}, PD) & \xrightarrow{\mathcal{C}(\mathbf{1}, Ph)} & \mathcal{C}(\mathbf{1}, PD') \\ \cong \downarrow & & \downarrow \cong \\ \widehat{\mathcal{D}}(D, O_{\mathcal{D}}) & \xrightarrow{\widehat{\mathcal{D}}(h, O_{\mathcal{D}})} & \widehat{\mathcal{D}}(D', O_{\mathcal{D}}) \end{array}$$

and similarly for  $|P^{-1}| \cong \mathcal{C}(-, O_{\mathcal{C}})$ . ◀

- **Remark C.1.** 1. By [19, Remark VI.2.4] the category  $\widehat{\mathcal{D}}$  of profinite  $\mathcal{D}$ -algebras is the pro-completion of  $\mathcal{D}_f$ . Therefore, by A.13, it is locally finitely copresentable, and its finitely copresentable objects are the objects of  $\mathcal{D}_f$ . For a *finite* set  $S$  of sorts, the category  $\mathcal{D}^S$  is locally finitely copresentable and its finitely copresentable objects are precisely the objects of  $\mathcal{D}_f^S$ . Hence, by A.13 again, the category  $\widehat{\mathcal{D}}^S$  is the pro-completion of  $\mathcal{D}_f^S$ .
2. The forgetful functor  $V : \widehat{\mathcal{D}} \rightarrow \mathcal{D}$  is a right adjoint and thus preserves limits, see [13, Proposition 2.8]. Since limits in  $\widehat{\mathcal{D}}^S$  and  $\mathcal{D}^S$  are computed sortwise, the same holds for the  $S$ -sorted forgetful functor  $V : \widehat{\mathcal{D}}^S \rightarrow \mathcal{D}^S$ .

**Details for Remark 2.12.2.** That epimorphisms in  $\widehat{\mathcal{D}}$  are surjective can be seen as follows.

1. First, observe that all monomorphisms in  $\widehat{\mathcal{D}}$  are injective because the right adjoint  $V : \widehat{\mathcal{D}} \rightarrow \mathcal{D}$ , see Remark C.1.2, preserves monomorphisms.
2. Next, we show that any epimorphism  $e : A \twoheadrightarrow B$  in  $\mathcal{D}_f$  is also an epimorphism in  $\mathcal{D}$  (and thus surjective by assumption). Suppose that  $f, g : B \rightarrow C$  are morphisms in  $\mathcal{D}$  with  $f \cdot e = g \cdot e$ . Express  $C$  as a directed union  $(c_i : C_i \rightarrow C)_{i \in I}$  of finite subobjects, using that  $\mathcal{D}$  is a locally finite variety. Since  $B$  is finite and the union is directed, the morphisms  $f$  and  $g$  factor through some  $c_i$ , i.e. there exist morphisms  $f', g'$  with  $f = c_i \cdot f'$  and  $g = c_i \cdot g'$ . Then

$$c_i \cdot f' \cdot e = f \cdot e = g \cdot e = c_i \cdot g' \cdot e,$$

and since  $c_i$  is monic, it follows that  $f' \cdot e = g' \cdot e$ . Since  $e$  is an epimorphism in  $\mathcal{D}_f$  and  $C_i$  is finite, this implies  $f' = g'$  and therefore  $f = g$ .

3. Now let  $e : A \twoheadrightarrow B$  be an arbitrary epimorphism in  $\widehat{\mathcal{D}}$ . By A.10, one can express  $e$  in the locally finitely copresentable category  $\widehat{\mathcal{D}}^{\rightarrow}$  as a cofiltered limit  $((a_i, b_i) : e \rightarrow f_i)_{i \in I}$  of morphisms  $f_i : A_i \rightarrow B_i$  in  $\mathcal{D}_f$ . Take the (epi, strong mono) factorizations of  $a_i$  and  $b_i$ , see A.14(c). Diagonal fill-in gives a morphism  $e_i$  as in the diagram below:

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ \downarrow a'_i & & \downarrow b'_i \\ A'_i & \xrightarrow{e_i} & B'_i \\ \downarrow a''_i & & \downarrow b''_i \\ A_i & \xrightarrow{f_i} & B_i \end{array} \quad \begin{array}{c} a_i \\ b_i \end{array}$$

The objects  $A'_i$  and  $B'_i$  are finite by part 1 of the proof. Moreover, since  $e$  and  $b'_i$  are epimorphic, so is  $e_i$ , and thus part 2 shows that  $e_i$  is surjective. Finally, observe that  $((a'_i, b'_i) : e \rightarrow e_i)_{i \in I}$  is a cofiltered limit cone in  $\widehat{\mathcal{D}}^{\rightarrow}$  by A.14(a),(c). Since limits in  $\widehat{\mathcal{D}}^{\rightarrow}$  are computed componentwise, Lemma B.1 shows that  $e$  is surjective. ◀

► **Remark C.2.** From the fact that  $\widehat{\mathcal{D}}$  is locally finitely copresentable and epimorphisms in  $\widehat{\mathcal{D}}$  are precisely the surjective morphisms, it follows that the factorization system (epi, strong mono) of  $\widehat{\mathcal{D}}$  coincides with (surjective, injective). Thus, dually,  $\mathcal{C}$  has the factorization system (strong epi, mono) = (surjective, injective).

► **Remark C.3.** We list some further properties of the profinite monad  $\widehat{\mathbf{T}}$ . See [13] for proofs.

1. For any  $\mathcal{D} \in \mathcal{D}_f^S$ , denote by  $(\mathbf{T}\mathcal{D} \downarrow \mathbf{Alg}_f \mathbf{T})$  the comma category of all  $\mathbf{T}$ -homomorphisms  $h : \mathbf{T}\mathcal{D} \rightarrow A$  with finite codomain, see A.4, and by  $\mathbf{Quo}_f(\mathbf{T}\mathcal{D})$  the full subcategory on *surjective* homomorphisms. The inclusion functor  $\mathbf{Quo}_f(\mathbf{T}\mathcal{D}) \hookrightarrow (\mathbf{T}\mathcal{D} \downarrow \mathbf{Alg}_f \mathbf{T})$  is final, cf. A.8. Therefore  $\widehat{\mathbf{T}}\mathcal{D}$ , see Construction 2.8, is also the cofiltered limit of the larger diagram

$$(\mathbf{T}\mathcal{D} \downarrow \mathbf{Alg}_f \mathbf{T}) \rightarrow \widehat{\mathcal{D}}^S, \quad (\mathbf{T}\mathcal{D} \xrightarrow{h} (A, \alpha)) \rightarrow A.$$

The limit projections are denoted by  $h^+ : \widehat{\mathbf{T}}\mathcal{D} \rightarrow A$ . The following squares commute for

all  $\mathbf{T}$ -homomorphisms  $h: \mathbf{T}D \rightarrow (A, \alpha)$  with  $(A, \alpha) \in \mathbf{Alg}_f \mathbf{T}$ :

$$\begin{array}{ccc}
 D & \xrightarrow{\hat{\eta}_D} & \hat{T}D \\
 & \searrow h\eta_D & \downarrow h^+ \\
 & & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 \hat{T}\hat{T}D & \xrightarrow{\hat{\mu}_D} & \hat{T}D \\
 \hat{T}h^+ \downarrow & & \downarrow h^+ \\
 \hat{T}A & \xrightarrow{\alpha^+} & A
 \end{array}
 \tag{2}$$

2. The profinite monad  $\hat{\mathbf{T}}$  is the codensity monad (see A.6) of the forgetful functor

$$\mathbf{Alg}_f \mathbf{T} \rightarrow \mathcal{D}_f^S \xrightarrow{\cong} \hat{\mathcal{D}}_f^S \rightarrow \hat{\mathcal{D}}^S.$$

The limit formula for right Kan extensions (see A.5) yields the construction of  $\hat{T}D$  and the commutative diagrams (2) in C.3.1.

3. Recall from Remark 2.11.1 the isomorphism  $\mathbf{Alg}_f \mathbf{T} \xrightarrow{\cong} \mathbf{Alg}_f \hat{\mathbf{T}}$ . Its inverse  $\mathbf{Alg}_f \hat{\mathbf{T}} \xrightarrow{\cong} \mathbf{Alg}_f \mathbf{T}$  is given by  $(B, \beta) \mapsto (B, V\beta \cdot \iota_B)$  and  $h \mapsto h$ . In the following we will often tacitly identify finite  $\mathbf{T}$ -algebras with their corresponding finite  $\hat{\mathbf{T}}$ -algebras.
4. Every finite  $\hat{\mathbf{T}}$ -algebra is finitely copresentable in  $\mathbf{Alg} \hat{\mathbf{T}}$ , see A.9.
5. Recall from Remark 2.11.2 the natural transformation  $\iota: TV \rightarrow V\hat{T}$ . Every  $\hat{\mathbf{T}}$ -homomorphism  $h: \hat{\mathbf{T}}D \rightarrow (A, \alpha^+)$  with  $(A, \alpha) \in \mathbf{Alg}_f \mathbf{T}$  and  $D \in \mathcal{D}_f^S$  restricts to a  $\mathbf{T}$ -homomorphism  $Vh \cdot \iota_D: \mathbf{T}D \rightarrow (A, \alpha)$ .
6. For any  $D \in \mathcal{D}_f^S$  the morphism  $\iota_D: TVD \rightarrow V\hat{T}D$  is dense, i.e. for each sort  $s$  the image of

$$\iota_D: (TVD)_s \rightarrow (V\hat{T}D)_s = V(\hat{T}D)_s$$

is a dense subset of the profinite  $\mathcal{D}$ -algebra  $(\hat{T}D)_s \in \hat{\mathcal{D}}$ . This implies that for any surjective morphism  $e: \hat{T}D \twoheadrightarrow A$  in  $\hat{\mathcal{D}}^S$  with  $A \in \hat{\mathcal{D}}_f^S$ , the restricted map  $Ve \cdot \iota_D: TD \twoheadrightarrow A$  is also surjective, as this map is dense and  $A$  is discrete. *We will use this property frequently.*

7. The functor  $\hat{T}$  preserves epimorphisms (= sortwise surjective morphisms) of  $\hat{\mathcal{D}}^S$ . Thus the factorization system of  $\mathcal{D}^S$  lifts to  $\mathbf{Alg} \hat{\mathbf{T}}$ : every  $\hat{\mathbf{T}}$ -homomorphism factorizes into a sortwise surjective homomorphism followed by a sortwise injective one.

► **Lemma C.4.** *Every  $\mathbf{T}$ -homomorphism  $g: \mathbf{T}D' \rightarrow \mathbf{T}D$  with  $D, D' \in \mathcal{D}_f^S$  extends uniquely to a  $\hat{\mathbf{T}}$ -homomorphism  $\hat{g}: \hat{\mathbf{T}}D' \rightarrow \hat{\mathbf{T}}D$  such that the following diagrams commute for all  $\mathbf{T}$ -homomorphisms  $h: \mathbf{T}D \rightarrow A$  with  $A \in \mathbf{Alg}_f \mathbf{T}$ :*

$$\begin{array}{ccc}
 TD' & \xrightarrow{g} & TD \\
 \downarrow \iota_{D'} & & \downarrow \iota_D \\
 V\hat{T}D' & \xrightarrow{V\hat{g}} & V\hat{T}D
 \end{array}
 \qquad
 \begin{array}{ccc}
 \hat{T}D' & \xrightarrow{\hat{g}} & \hat{T}D \\
 (hg)^+ \searrow & & \downarrow h^+ \\
 & & A
 \end{array}
 \tag{3}$$

**Proof.** The morphisms  $(hg)^+$  form a compatible family over the limit cone defining  $\hat{T}D$ , i.e. for all  $\mathbf{T}$ -homomorphisms  $k: A \rightarrow A'$  in  $\mathbf{Alg}_f \mathbf{T}$  we have  $(khg)^+ = k \cdot (hg)^+$ . Indeed, this



holds when precomposed with the dense map  $\iota_{D'}$ , as shown the commutative diagram below:

$$\begin{array}{ccc}
 TD' & \xrightarrow{g} & D \\
 \downarrow \iota_{D'} & & \downarrow \iota_D \\
 V\hat{T}D' & \xrightarrow{V\hat{g}} & V\hat{T}D \\
 \downarrow V(hg)^+ & & \downarrow Vh^+ \\
 & & A \\
 & & \downarrow k \\
 & & A'
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow kh \\
 \searrow V(kh)^+ \\
 \nearrow V(khg)^+
 \end{array}$$

Thus there exists a unique  $\hat{g} : \hat{T}D' \rightarrow \hat{T}D$  with  $(hg)^+ = h^+ \cdot \hat{g}$  for all  $h$ , i.e. the right diagram of (3) commutes. This also implies that the left diagram commutes. Indeed, it commutes when postcomposed with the morphisms  $Vh^+$ , and the latter are jointly monomorphic because  $V$  preserves limits, see Remark C.1.2. ◀

### D Details for Section 3

**Proof of Theorem 3.3.** We first show that the language  $L := V\hat{L} \cdot \iota_{\Sigma} : T\Sigma \rightarrow O_{\mathcal{D}}$  is recognizable for any morphism  $\hat{L} : \hat{T}\Sigma \rightarrow O_{\mathcal{D}}$  in  $\hat{\mathcal{D}}^S$ . Since  $O_{\mathcal{D}}$  is finitely copresentable in  $\hat{\mathcal{D}}^S$ , see Remark C.1, the morphism  $\hat{L}$  factors through the cofiltered limit cone defining  $\hat{T}\Sigma$ , i.e. there exists a  $\mathbf{T}$ -homomorphism  $h : \mathbf{T}\Sigma \rightarrow A$  with  $A \in \mathbf{Alg}_f \mathbf{T}$  and a morphism  $p : A \rightarrow O_{\mathcal{D}}$  in  $\hat{\mathcal{D}}^S$  with  $\hat{L} = p \cdot h^+$ . It follows that  $L$  is recognized by  $h$  via  $p$ , see the diagram below:

$$\begin{array}{ccc}
 T\Sigma & \xrightarrow{L} & O_{\mathcal{D}} \\
 \downarrow \iota_{\Sigma} & \searrow h & \uparrow p=Vp \\
 V\hat{T}\Sigma & \xrightarrow{Vh^+} & A
 \end{array} \tag{4}$$

Conversely, let  $L : T\Sigma \rightarrow O_{\mathcal{D}}$  be any recognizable language. Choose a  $\mathbf{T}$ -homomorphism  $h : \mathbf{T}\Sigma \rightarrow A$  with  $A$  finite and a morphism  $p : A \rightarrow O_{\mathcal{D}}$  with  $L = p \cdot h$ . This yields the following morphism in  $\hat{\mathcal{D}}^S$ :

$$\hat{L} = (\hat{T}\Sigma \xrightarrow{h^+} A \xrightarrow{p} O_{\mathcal{D}}).$$

Since  $L = V\hat{L} \cdot \iota_{\Sigma}$  and  $\iota_{\Sigma}$  is dense by Remark C.3.6, the morphism  $\hat{L}$  is independent of the choice of  $h$  and  $p$ . Clearly the maps  $\hat{L} \mapsto L$  and  $L \mapsto \hat{L}$  are mutually inverse, which proves the claim. ◀

**Details for Remark 3.4.** We verify that  $\text{Rec}(\Sigma)$ , equipped with the  $\mathcal{C}$ -algebraic structure isomorphic to  $\prod_s P(\hat{T}\Sigma)_s$ , is a subobject of  $\prod_s O_{\mathcal{C}}^{|T\Sigma|_s}$  via the map

$$(T\Sigma \xrightarrow{L} O_{\mathcal{D}}) \mapsto (|T\Sigma|_s \xrightarrow{|L|} |O_{\mathcal{D}}| \xrightarrow{\cong} |O_{\mathcal{C}}|)_{s \in S}$$

1. We first show that, for each sort  $s$ , the object  $P(\hat{T}\Sigma)_s$  forms a subobject of  $O_{\mathcal{C}}^{|T\Sigma|_s}$ . For each element  $x : \mathbb{1} \rightarrow (T\Sigma)_s$  of  $|T\Sigma|_s$ , the  $\mathcal{D}$ -morphism  $(\mathbb{1} \xrightarrow{x} (T\Sigma)_s \xrightarrow{\iota_{\Sigma}} V(\hat{T}\Sigma)_s)$  is continuous, because  $\mathbb{1}$  is finite and thus discrete. That is, there exists a morphism  $\hat{x} : \mathbb{1} \rightarrow (\hat{T}\Sigma)_s$  in  $\hat{\mathcal{D}}$  with  $V\hat{x} = \iota_{\Sigma} \cdot x$ . Since the morphisms  $x$  are jointly surjective, and  $\iota_{\Sigma}$

is dense by Remark C.3.6, the family  $(\hat{x})_{x: \mathbb{1} \rightarrow (T\Sigma)_s}$  forms a jointly epimorphic family in  $\hat{\mathcal{D}}$ . Thus the dual family  $(P\hat{x}: P(\hat{T\Sigma})_s \rightarrow O_{\mathcal{C}})$  in  $\mathcal{C}$  is jointly monomorphic, which implies that the induced morphism  $m_s$  into the product (making the triangle below commute for all  $x$ ) is monomorphic.

$$\begin{array}{ccc} P(\hat{T\Sigma})_s & \xrightarrow{m_s} & O_{\mathcal{C}}^{|T\Sigma|_s} \\ & \searrow P\hat{x}' & \downarrow \pi_x \\ & & O_{\mathcal{C}} \end{array}$$

2. It follows that  $\text{Rec}(\Sigma)$  is a subobject of  $\prod_s O_{\mathcal{C}}^{|T\Sigma|_s}$  via the embedding

$$\text{Rec}(\Sigma) \cong \hat{\mathcal{D}}^S(\hat{T\Sigma}, O_{\mathcal{D}}) \cong \prod_s \hat{\mathcal{D}}((\hat{T\Sigma})_s, O_{\mathcal{D}}) \cong \prod_s |P(\hat{T\Sigma})_s| \xrightarrow{\prod_s m_s} \prod_s O_{\mathcal{C}}^{|T\Sigma|_s}.$$

By applying the definitions of the three bijections and of the morphisms  $m_s$ , one easily verifies that this embedding maps a recognizable language  $L: T\Sigma \rightarrow O_{\mathcal{D}}$  to the element  $(|T\Sigma|_s \xrightarrow{|L|} |O_{\mathcal{D}}| \xrightarrow{\cong} |O_{\mathcal{C}}|)_{s \in S}$  of  $\prod_s O_{\mathcal{C}}^{|T\Sigma|_s}$ , as claimed.  $\blacktriangleleft$

**Details for Example 3.6.3.** Every polynomial  $p: 1_{s'} \rightarrow T(\Sigma + 1_s)$  induces an evaluation map  $[p]: (T\Sigma)_{s'} \rightarrow (T\Sigma)_{s'}$  that sends an element  $x: 1_s \rightarrow T\Sigma$  of  $(T\Sigma)_s$  to the following element of  $(T\Sigma)_{s'}$ :

$$1_{s'} \xrightarrow{p} T(\Sigma + 1_s) \xrightarrow{T(\Sigma+x)} T(\Sigma + T\Sigma) \xrightarrow{T(\eta+T\Sigma)} T(T\Sigma + T\Sigma) \xrightarrow{T[id, id]} TT\Sigma \xrightarrow{\mu_\Sigma} T\Sigma.$$

► **Lemma D.1.** For any object  $D \in \mathcal{D}_f^S$  and any two elements  $x, y \in |D|_s$  with  $s \in S$  we have

$$x = y \quad \text{iff} \quad \forall (D \xrightarrow{k} O_{\mathcal{D}}) : k(x) = k(y).$$

**Proof.** Given  $x \neq y \in |D|_s$  we need to find a morphism  $k: D \rightarrow O_{\mathcal{D}}$  with  $k(x) \neq k(y)$ . Since  $O_{\mathcal{D}}$  is a cogenerator of  $\mathcal{D}_f$  (being dual to the generator  $\mathbb{1}$  of  $\mathcal{C}_f$ ), there exists a morphism  $k_s: D_s \rightarrow O_{\mathcal{D}}$  in  $\mathcal{D}$  with  $k_s(x) \neq k_s(y)$ . For any sort  $t \neq s$ , pick an arbitrary morphism  $k_t: D_t \rightarrow O_{\mathcal{D}}$ . Such a morphism exists because, by our Assumption 2.1 that the signature of  $\mathcal{C}$  has a constant, we have dually a morphism  $\mathbb{1} \rightarrow PD_t$  in  $\mathcal{C}_f$ . Thus  $k: D \rightarrow O_{\mathcal{D}}$  is a morphism in  $\mathcal{D}^S$  with  $k(x) \neq k(y)$ .  $\blacktriangleleft$

**Proof of Theorem 3.9.** We will repeatedly use the *homomorphism theorem*: given  $e: A \rightarrow B$  and  $f: A \rightarrow C$  in  $\mathcal{D}^S$  with  $e$  epimorphic, there exists a morphism  $g$  with  $g \cdot e = f$  iff, for all sorts  $s$  and  $a, a' \in |A|_s$ ,  $e(a) = e(a')$  implies  $f(a) = f(a')$ . Put  $A_L := \mathbf{T\Sigma}/\equiv_L$ .

**“only if” direction.** Suppose that  $\mathbb{U}_\Sigma$  is a unary presentation of  $\mathbf{T}$  over  $\Sigma$ , and let  $L: T\Sigma \rightarrow O_{\mathcal{D}}$  be a recognizable language.

- (a) We show that there exists a morphism  $p_L: A_L \rightarrow O_{\mathcal{D}}$  in  $\mathcal{D}^S$  with  $L = p_L \cdot e_L$ , using the homomorphism theorem. Let  $x, y \in |T\Sigma|_s$  with  $e_L(x) = e_L(y)$ , i.e.  $x \equiv_L y$ . Since  $\mathbb{U}_\Sigma$  contains all identities, putting  $s' := s$  and  $u := id_{(T\Sigma)_s}$  in the definition of  $\equiv_L$  (see Definition 3.8) yields  $L(x) = L(y)$ . The homomorphism theorem gives the desired  $p_L$ .

- (b) Since  $L$  is recognizable, there is a surjective  $\mathbf{T}$ -homomorphism  $e : T\mathbb{Z} \twoheadrightarrow A$  into a finite  $\mathbf{T}$ -algebra  $A$  and a morphism  $p : A \rightarrow O_{\mathcal{D}}$  in  $\mathcal{D}^S$  with  $L = p \cdot e$ . Furthermore, since  $\mathbb{U}_{\Sigma}$  forms a unary presentation, we can choose for each  $u : (T\mathbb{Z})_s \rightarrow (T\mathbb{Z})_{s'}$  in  $\mathbb{U}_{\Sigma}$  a lifting  $u_A : A_s \rightarrow A_{s'}$  along  $e$ , that is,  $e \cdot u = u_A \cdot e$ . We claim that there exists a morphism  $h : A \rightarrow A_L$  in  $\mathcal{D}^S$  with  $e_L = h \cdot e$ . This follows from the homomorphism theorem: let  $x, y \in |T\mathbb{Z}|_s$  with  $e(x) = e(y)$ . Then, for all sorts  $s'$  and  $u : (T\mathbb{Z})_s \rightarrow (T\mathbb{Z})_{s'}$  in  $\mathbb{U}_{\Sigma}$ ,

$$L \cdot u(x) = p \cdot e \cdot u(x) = p \cdot u_A \cdot e(x) = p \cdot u_A \cdot e(y) = p \cdot e \cdot u(y) = L \cdot u(y).$$

Thus  $x \equiv_L y$ , or equivalently  $e_L(x) = e_L(y)$ , and the homomorphism theorem gives the desired  $h$ .

- (c) We show that (I)  $e_L$  is extensible, and (II) every morphism  $u : (T\mathbb{Z})_s \rightarrow (T\mathbb{Z})_{s'}$  in  $\mathbb{U}_{\Sigma}$  has a lifting along  $e_L$ . This implies the claim: since  $\mathbb{U}_{\Sigma}$  is a unary presentation,  $A_L$  then carries a  $\mathbf{T}$ -algebra structure making  $e_L$  a  $\mathbf{T}$ -homomorphism. And part (a) and (b) show that  $e_L$  recognizes  $L$  and has the universal property of a syntactic morphism.

For (I), the following commutative diagram shows that  $e_L$  is extensible:

$$\begin{array}{ccccc} & & T\mathbb{Z} & & \\ & \swarrow \iota_{\mathbb{Z}} & \downarrow e & \searrow e_L & \\ V\hat{T}\mathbb{Z} & \xrightarrow[V_{e^+}]{} & A & \xrightarrow{h} & A_L \end{array}$$

Indeed, the left square commutes by Remark 2.11.2, and the right square by (b). Thus  $e_L$  has the continuous extension  $h \cdot e^+$ .

For (II), by the homomorphism theorem we need to show that for all  $x, y \in |T\mathbb{Z}|_s$  with  $x \equiv_L y$  we have  $u(x) \equiv_L u(y)$ . Note that for all sorts  $s''$  and all  $u' : (T\mathbb{Z})_{s'} \rightarrow (T\mathbb{Z})_{s''}$  in  $\mathbb{U}_{\Sigma}$  we have  $u' \cdot u \in \mathbb{U}_{\Sigma}$  because  $\mathbb{U}_{\Sigma}$  is closed under composition. Thus  $x \equiv_L y$  implies

$$L \cdot (u' \cdot u)(x) = L \cdot (u' \cdot u)(y) \quad \text{for all sorts } s'' \text{ and } u' : (T\mathbb{Z})_{s'} \rightarrow (T\mathbb{Z})_{s''} \text{ in } \mathbb{U}_{\Sigma},$$

which means precisely that  $u(x) \equiv_L u(y)$ .

**“if” direction.** Suppose that, for any recognizable language  $L$  over  $\Sigma$ , the morphism  $e_L$  is a  $\mathbf{T}$ -algebra congruence, and moreover  $e_L : T\mathbb{Z} \twoheadrightarrow A_L$  is a syntactic morphism of  $L$ . We verify that  $\mathbb{U}_{\Sigma}$  is a unary presentation, i.e. the equivalence of (i) and (ii) in Definition 3.7 for any extensible finite quotient  $e : T\mathbb{Z} \twoheadrightarrow A$  in  $\mathcal{D}^S$ .

(i) $\Rightarrow$ (ii) Suppose that  $A$  carries a  $\mathbf{T}$ -algebra structure making  $e : T\mathbb{Z} \twoheadrightarrow A$  a  $\mathbf{T}$ -homomorphism. We need to show that every morphism  $u : (T\mathbb{Z})_s \rightarrow (T\mathbb{Z})_{s'}$  in  $\mathbb{U}_{\Sigma}$  has a lifting along  $e$ , i.e. there exists a morphism  $u_A : A_s \rightarrow A_{s'}$  with  $e \cdot u = u_A \cdot e$ . This requires another use of the homomorphism theorem. For any morphism  $k : A \rightarrow O_{\mathcal{D}}$  in  $\mathcal{D}^S$  we have the recognizable language  $L_k := k \cdot e : T\mathbb{Z} \rightarrow O_{\mathcal{D}}$ , which by hypothesis has the syntactic morphism  $e_{L_k} : T\mathbb{Z} \twoheadrightarrow A_{L_k}$ . Since  $e_{L_k}$  recognizes  $L_k$ , there exists a morphism  $p_{L_k} : A_{L_k} \rightarrow O_{\mathcal{D}}$  with  $L_k = p_{L_k} \cdot e_{L_k}$ . Furthermore, the universal property of the syntactic morphism  $e_{L_k}$  gives a unique  $\mathbf{T}$ -homomorphism  $h_k : A \twoheadrightarrow A_L$  with  $e_{L_k} = h_k \cdot e$ . Thus for all  $x, y \in |T\mathbb{Z}|_s$  we have the following implications:

$$\begin{aligned} e(x) = e(y) &\Rightarrow \forall (k : A \rightarrow O_{\mathcal{D}}) : e_{L_k}(x) = e_{L_k}(y) && (e_{L_k} = h_k \cdot e) \\ &\Leftrightarrow \forall k : x \equiv_{L_k} y && (\text{def. } e_{L_k}) \\ &\Rightarrow \forall k : L_k \cdot u(x) = L_k \cdot u(y) && (\text{def. } \equiv_{L_k}) \\ &\Leftrightarrow \forall k : k \cdot e \cdot u(x) = k \cdot e \cdot u(y) && (\text{def. } L_k) \\ &\Leftrightarrow e \cdot u(x) = e \cdot u(y) && (\text{Lemma D.1}). \end{aligned}$$

Thus the homomorphism theorem gives the desired lifting  $u_A$ .

(ii) $\Rightarrow$ (i) Let  $e = V\hat{e} \cdot \iota_{\Sigma} : T\Sigma \twoheadrightarrow A$  be an extensible finite quotient in  $\mathcal{D}^S$ , and suppose that every  $u : (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbb{U}_{\Sigma}$  has a lifting  $u_A : A_s \rightarrow A_{s'}$  along  $e$ . We need to show that  $A$  carries a  $\mathbf{T}$ -algebra structure making  $e$  a  $\mathbf{T}$ -homomorphism. For each  $k : A \rightarrow O_{\mathcal{D}}$  in  $\mathcal{D}^S$  the language  $L_k := k \cdot e : T\Sigma \rightarrow O_{\mathcal{D}}$  is recognizable by Theorem 3.3, since  $L_k = V(k \cdot \hat{e}) \cdot \iota_{\Sigma}$ . Thus by hypothesis we have the syntactic morphism  $e_{L_k} : \mathbf{T}\Sigma \twoheadrightarrow A_{L_k}$ . Since  $e_{L_k}$  recognizes  $L_k$ , there exists a morphism  $p_{L_k} : A_{L_k} \rightarrow O_{\mathcal{D}}$  with  $L_k = p_{L_k} \cdot e_{L_k}$ .

We claim that  $e_{L_k}$  factors through  $e$ . To see this, we use the homomorphism theorem. Given  $x, y \in |T\Sigma|_s$  with  $e(x) = e(y)$ , we have for all  $u : (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbb{U}_{\Sigma}$ :

$$\begin{aligned} L_k \cdot u(x) &= k \cdot e \cdot u(x) && (\text{def. } L_k) \\ &= k \cdot u_A \cdot e(x) && (\text{def. } u_A) \\ &= k \cdot u_A \cdot e(y) && (e(x) = e(y)) \\ &= k \cdot e \cdot u(y) && (\text{def. } u_A) \\ &= L_k \cdot u(y) && (\text{def. } L_k). \end{aligned}$$

Thus  $x \equiv_{L_k} y$ , or equivalently  $e_{L_k}(x) = e_{L_k}(y)$ . The homomorphism theorem yields a morphism  $h_k$  with  $e_{L_k} = h_k \cdot e$ .

We are ready to define the desired  $\mathbf{T}$ -algebra structure on  $A$  for which  $e$  is a  $\mathbf{T}$ -homomorphism. It suffices to find a morphism  $\alpha : TA \rightarrow A$  in  $\mathcal{D}^S$  making the following square commute:

$$\begin{array}{ccc} TT\Sigma & \xrightarrow{\mu_{\Sigma}} & T\Sigma \\ Te \downarrow & & \downarrow e \\ TA & \xrightarrow{\alpha} & A \end{array}$$

Since  $T$  preserves epimorphisms, this then immediately implies that  $(A, \alpha)$  is a  $\mathbf{T}$ -algebra and  $e$  is a  $\mathbf{T}$ -homomorphism. To construct  $\alpha$  we once again use the homomorphism theorem. The proof is illustrated by the diagram below, where  $\alpha_{L_k}$  is the  $\mathbf{T}$ -algebra structure of  $A_{L_k}$ .

$$\begin{array}{ccccc} TT\Sigma & \xrightarrow{\mu_{\Sigma}} & T\Sigma & & \\ Te \downarrow & & \downarrow e & \searrow L_k & \\ TA & \xrightarrow{\alpha} & A & \xrightarrow{k} & O_{\mathcal{D}} \\ Th_k \downarrow & & \downarrow h_k & \nearrow p_{L_k} & \\ TA_{L_k} & \xrightarrow{\alpha_{L_k}} & A_{L_k} & & \end{array}$$

For all  $x, y \in |TT\Sigma|_s$  with  $Te(x) = Te(y)$  we have

$$\begin{aligned} k \cdot e \cdot \mu_{\Sigma}(x) &= L_k \cdot \mu_{\Sigma}(x) && (\text{def. } L_k) \\ &= p_{L_k} \cdot e_{L_k} \cdot \mu_{\Sigma}(x) && (\text{def. } p_{L_k}, e_{L_k}) \\ &= p_{L_k} \cdot \alpha_{L_k} \cdot Te_{L_k}(x) && (e_{L_k} \text{ is } \mathbf{T}\text{-hom.}) \\ &= p_{L_k} \cdot \alpha_{L_k} \cdot Th_k \cdot Te(x) && (\text{def. } h_k) \\ &= p_{L_k} \cdot \alpha_{L_k} \cdot Th_k \cdot Te(y) && (Te(x) = Te(y)) \\ &= \dots && (\text{compute backwards}) \\ &= k \cdot e \cdot \mu_{\Sigma}(y). \end{aligned}$$

Since this holds for all  $k : A \rightarrow O_{\mathcal{D}}$ , Lemma D.1 implies that  $e \cdot \mu_{\Sigma}(x) = e \cdot \mu_{\Sigma}(y)$ . Thus the homomorphism theorem yields the desired **T**-algebra structure  $\alpha$ . ◀

**Details for Remark 3.12.** Recall that for a finitary signature  $\Gamma$ , an *ordered  $\Gamma$ -algebra* is a poset equipped with monotone  $\Gamma$ -operations. Morphisms of ordered algebras are monotone maps preserving all  $\Gamma$ -operations. A *variety of ordered algebras* is a class of ordered  $\Gamma$ -algebras closed under quotients, subalgebras, and products. Here *quotients* are represented by surjective morphisms, and *subalgebras* by order-reflecting morphisms (i.e.  $mx \leq my$  iff  $x \leq y$ ). Varieties of ordered algebras are precisely the classes of ordered algebras that can be specified by inequations  $s \leq t$  between  $\Gamma$ -terms. This ordered analogue of Birkhoff’s variety theorem is due to Bloom [11].

In Assumptions 2.1 one can make the more general assumption that  $\mathcal{D}$  is either a locally finite variety of algebras or a locally finite variety of ordered algebras. All constructions, theorems, and proofs in Section 2, 4 and 5 are identical for the ordered case, except that Stone spaces need to be replaced by Priestley spaces. In Section 3 we make the following adaptations: first,  $\equiv_L$  is replaced by the syntactic preorder  $\leq_L$  (see Remark 2.1). Second, Lemma D.1 is replaced by

**Lemma D.1’.** Let  $\mathcal{D}$  be a variety of ordered algebras and suppose that  $O_{\mathcal{D}}$  is an order-cogenerator, see Remark 3.12. Then for any object  $D \in \mathcal{D}_f^S$  and any two elements  $x, y \in |D|_s$  with  $s \in S$  we have

$$x \leq y \quad \text{iff} \quad \forall (D \xrightarrow{k} O_{\mathcal{D}}) : k(x) \leq k(y).$$

The proof of this lemma is completely analogous to the one of Lemma D.1. Finally, for Theorem 3.9 to hold, one needs to assume for the “only if” direction that  $O_{\mathcal{D}}$  is an order-cogenerator. In the above proof of the theorem, one replaces equations by inequations, invokes Lemma D.1’ in lieu of D.1, and uses the homomorphism theorem for ordered algebras: given morphisms  $e : A \twoheadrightarrow B$  and  $f : A \rightarrow C$  in  $\mathcal{D}^S$  with  $e$  epimorphic, there exists a morphism  $g$  with  $g \cdot e = f$  iff, for all sorts  $s$  and  $a, a' \in |A|_s$ ,  $e(a) \leq e(a')$  implies  $f(a) \leq f(a')$ .

Let us emphasize that the sole relevance of Theorem 3.9 is for the construction of examples. The theoretical results in Section 4 and 5 are not depending on this theorem and hold, as in the unordered case, without any assumptions on  $O_{\mathcal{D}}$ . This holds, in particular, for our main result (Theorem 5.9). ◀

## E Details for Section 4

► **Remark E.1.** Every free  $\widehat{\mathbf{T}}$ -algebra  $\widehat{\mathbf{T}}D$  with  $D \in \mathcal{D}_f^S$  is profinite. Indeed, since the forgetful functor from  $\mathbf{Alg} \widehat{\mathbf{T}}$  to  $\widehat{\mathcal{D}}^S$  reflects limits, see A.3, the right square of (2) shows that the  $\widehat{\mathbf{T}}$ -algebra  $\widehat{\mathbf{T}}D$  is the cofiltered limit of the diagram

$$(\mathbf{T}D \downarrow \mathbf{Alg}_f \mathbf{T}) \rightarrow \mathbf{Alg} \widehat{\mathbf{T}}, \quad (h : \widehat{\mathbf{T}}D \rightarrow (A, \alpha)) \mapsto (A, \alpha^+),$$

with limit projections  $h^+$ .

► **Lemma E.2.** A  $\widehat{\mathbf{T}}$ -algebra  $A$  is profinite iff  $A$  is the limit of the cofiltered diagram

$$(A \downarrow \mathbf{Alg}_f \widehat{\mathbf{T}}) \rightarrow \mathbf{Alg} \widehat{\mathbf{T}}, \quad (h : A \rightarrow A') \mapsto A',$$

with limit projections  $h : A \rightarrow A'$ .

**Proof.** The “if” direction is trivial. For the “only if” direction, suppose that  $A$  is profinite, i.e. there exists a cofiltered limit cone  $(p_i : A \rightarrow A_i)$  in  $\mathbf{Alg} \hat{\mathbf{T}}$  with  $A_i \in \mathbf{Alg}_f \hat{\mathbf{T}}$ . Since the forgetful functor  $\hat{U} : \mathbf{Alg} \hat{\mathbf{T}} \rightarrow \hat{\mathcal{D}}^S$  reflects limits, it suffices to show that the cofiltered cone  $(h : \hat{U}A \rightarrow \hat{U}A')$  in  $\hat{\mathcal{D}}^S$  is a limit cone. To this end we verify the criterion of A.11. For (i), let  $f : \hat{U}A \rightarrow B$  be a morphism in  $\hat{\mathcal{D}}^S$  with  $B \in \hat{\mathcal{D}}_f^S$ . Since  $\hat{U}$  preserves limits, we have the limit cone  $(p_i : \hat{U}A \rightarrow \hat{U}A_i)$  in  $\hat{\mathcal{D}}^S$ . Moreover, since  $B$  is finitely copresentable in  $\hat{\mathcal{D}}^S$ , see Remark C.1, there exists an  $i$  and morphism  $f' : \hat{U}A_i \rightarrow B$  with  $f = f' \cdot p_i$ . This proves that  $f$  factors through the cone  $(h : \hat{U}A \rightarrow \hat{U}A')$  via  $h = p_i$  and  $f'$ , as desired.

For (ii), suppose that  $h : A \rightarrow A'$  in  $(A \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  and  $f', f'' : \hat{U}A' \rightarrow B$  are given with  $f' \cdot h = f'' \cdot h$ . Since  $A'$  is finitely copresentable in  $\mathbf{Alg} \hat{\mathbf{T}}$  by Remark 2.11.4, there exists an  $i$  and a  $\hat{\mathbf{T}}$ -homomorphism  $h' : A_i \rightarrow A'$  with  $h = h' \cdot p_i$ . Then  $(f' \cdot h') \cdot p_i = (f'' \cdot h') \cdot p_i$ . Thus, by (ii) applied to the cofiltered limit cone  $(p_i : \hat{U}A \rightarrow \hat{U}A_i)$ , we have a morphism  $a_{ji} : A_j \rightarrow A_i$  in the diagram with  $f' \cdot h' \cdot a_{ji} = f'' \cdot h' \cdot a_{ji}$ . Thus  $h' \cdot a_{ji} : p_j \rightarrow h$  is a morphism in  $(A \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  that merges  $f'$  and  $f''$ , as desired.

$$\begin{array}{ccccc}
 & & \hat{U}A & & \\
 & \swarrow p_j & \downarrow h & \searrow f & \\
 \hat{U}A_j & \xleftarrow{a_{ji}} & \hat{U}A_i & \xrightarrow{h'} & \hat{U}A' \xrightarrow[f'']{f'} B
 \end{array}$$

◀

► **Notation E.3.** Let  $(A \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  be the full subcategory of  $(A \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  on all surjective  $\hat{\mathbf{T}}$ -homomorphisms  $e : A \twoheadrightarrow A'$  with finite codomain.

► **Corollary E.4.** A  $\hat{\mathbf{T}}$ -algebra  $A$  is profinite iff  $A$  is the limit of the cofiltered diagram

$$(A \downarrow \mathbf{Alg}_f \hat{\mathbf{T}}) \rightarrow \mathbf{Alg} \hat{\mathbf{T}}, \quad (e : A \twoheadrightarrow A') \mapsto A',$$

with limit projections  $e : A \twoheadrightarrow A'$ .

**Proof.** By Remark C.3.7,  $(A \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  is a final cofiltered subcategory of  $(A \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$ . ◀

► **Lemma E.5.** Let  $\mathbb{U}_\Sigma$  be a unary presentation of  $\mathbf{T}$  over  $\Sigma$ , and let  $e : \mathbf{T}\Sigma \twoheadrightarrow A$  and  $k : A \twoheadrightarrow B$  be surjective  $\mathbf{T}$ -homomorphisms with  $A, B \in \mathbf{Alg}_f \mathbf{T}$ . Then the following diagram commutes for all  $u : (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbb{U}_\Sigma$ , where  $u_A$  and  $u_B$  are the liftings of  $u$  along  $e$  and  $k \cdot e$ , respectively.

$$\begin{array}{ccccc}
 (T\Sigma)_s & \xrightarrow{e} & A_s & \xrightarrow{k} & B_s \\
 u \downarrow & & \downarrow u_A & & \downarrow u_B \\
 (T\Sigma)_{s'} & \xrightarrow{e} & A_{s'} & \xrightarrow{k} & B_{s'}
 \end{array}$$

**Proof.** Clear since  $e$  is an epimorphism. ◀

► **Lemma E.6.** Let  $\mathbb{U}_\Sigma$  be a unary presentation of  $\mathbf{T}$  over  $\Sigma$ . Then every  $u : (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbb{U}_\Sigma$  has a unique extension to a morphism  $\hat{u} : (\hat{T}\Sigma)_s \rightarrow (\hat{T}\Sigma)_{s'}$  in  $\hat{\mathcal{D}}$  making the following square commute.

$$\begin{array}{ccc}
 (T\Sigma)_s & \xrightarrow{u} & (T\Sigma)_{s'} \\
 \downarrow \iota_\Sigma & & \downarrow \iota_\Sigma \\
 V(\hat{T}\Sigma)_s & \xrightarrow{V\hat{u}} & V(\hat{T}\Sigma)_{s'}
 \end{array}$$



**Proof.** For each  $u : (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbb{U}_\Sigma$ , the morphisms  $u_A \cdot e^+ : (\hat{T}\Sigma)_s \rightarrow A_{s'}$  (where  $e$  ranges over surjective  $\mathbf{T}$ -homomorphisms  $e : T\Sigma \twoheadrightarrow A$  with  $A \in \mathbf{Alg}_f \mathbf{T}$  and  $u_A$  is the lifting of  $u$  along  $e$ ) form a compatible family over the diagram defining  $(\hat{T}\Sigma)_{s'}$  by Lemma E.5. Hence there exists a unique morphism  $\hat{u} : (\hat{T}\Sigma)_s \rightarrow (\hat{T}\Sigma)_{s'}$  in  $\hat{\mathcal{D}}$  with  $e^+ \cdot \hat{u} = u_A \cdot e^+$  for all  $e$ . Therefore in the diagram below the outer square and all parts except for the upper square commute:

$$\begin{array}{ccc}
 (T\Sigma)_s & \xrightarrow{u} & (T\Sigma)_{s'} \\
 \downarrow \iota_\Sigma & & \downarrow \iota_\Sigma \\
 V(\hat{T}\Sigma)_s & \xrightarrow{V\hat{u}} & V(\hat{T}\Sigma)_{s'} \\
 \downarrow Ve^+ & & \downarrow Ve^+ \\
 A_s & \xrightarrow{u_A} & A_{s'}
 \end{array}
 \quad \begin{array}{c}
 \curvearrowright e \\
 \curvearrowleft e
 \end{array}$$

It follows that the upper square commutes when postcomposed with the morphisms  $Ve^+$ . Since by Remark C.1 the functor  $V$  preserves limits (and thus the morphisms  $Ve^+$  are jointly monomorphic), the upper square commutes. Moreover,  $\hat{u}$  is unique with this property because  $\iota_\Sigma$  is dense (see Remark 2.11.2) and  $A_{s'}$  is a Hausdorff space.  $\blacktriangleleft$

► **Remark E.7.** It follows that, for any extensible morphism  $e = V\hat{e} \cdot \iota_\Sigma : T\Sigma \twoheadrightarrow A$ , a morphism  $u_A : A \rightarrow A$  is a lifting of  $u : (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  along  $e$  iff it is a lifting of  $\hat{u}$  along  $\hat{e}$ , i.e. iff the following square commutes:

$$\begin{array}{ccc}
 (\hat{T}\Sigma)_s & \xrightarrow{\hat{u}} & (\hat{T}\Sigma)_{s'} \\
 \downarrow \hat{e} & & \downarrow \hat{e} \\
 A_s & \xrightarrow{u_A} & A_{s'}
 \end{array}$$

The following lemma shows that the lifting property of a unary presentation extends from finite to profinite algebras:

► **Lemma E.8.** *Let  $\mathbb{U}_\Sigma$  be a unary presentation of  $\mathbf{T}$  over  $\Sigma$ . Then for any epimorphism  $\hat{e} : \hat{T}\Sigma \twoheadrightarrow A$  in  $\hat{\mathcal{D}}^S$  the following statements are equivalent:*

- (i) *There exists a  $\hat{\mathbf{T}}$ -algebra structure on  $A$  making  $\hat{e} : \hat{\mathbf{T}}\Sigma \rightarrow A$  a  $\Sigma$ -generated profinite  $\hat{\mathbf{T}}$ -algebra.*
- (ii) *Each  $u : (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbb{U}_\Sigma$  has a lifting along  $\hat{e}$ , i.e. there exists a morphism  $u_A : A_s \rightarrow A_{s'}$  in  $\hat{\mathcal{D}}$  for which the following square commutes:*

$$\begin{array}{ccc}
 (\hat{T}\Sigma)_s & \xrightarrow{\hat{u}} & (\hat{T}\Sigma)_{s'} \\
 \downarrow \hat{e} & & \downarrow \hat{e} \\
 A_s & \xrightarrow{u_A} & A_{s'}
 \end{array} \tag{5}$$

**Proof.** (i)  $\Rightarrow$  (ii) Let  $\hat{e} : \hat{\mathbf{T}}\Sigma \twoheadrightarrow A$  be a  $\Sigma$ -generated profinite  $\hat{\mathbf{T}}$ -algebra. For any finite quotient algebra  $h : A \twoheadrightarrow A'$  in  $\mathbf{Alg}_f \hat{\mathbf{T}}$  we have the surjective  $\mathbf{T}$ -homomorphism  $e := V(h \cdot \hat{e}) \cdot \iota_\Sigma : T\Sigma \twoheadrightarrow A'$ , see Remark 2.11.5. Since  $\mathbb{U}_\Sigma$  is a unary presentation, each  $u : (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbb{U}_\Sigma$  has a lifting  $u_{A'} : A'_s \rightarrow A'_{s'}$  along  $e$ .

Since  $A$  is profinite,  $A$  is the cofiltered limit of the diagram of all finite quotients  $h : A \twoheadrightarrow A'$ , see Lemma E.2. The morphisms  $u_{A'} \cdot h : A_s \rightarrow A'_{s'}$  form a compatible family

over this diagram by Lemma E.5. Therefore there exists a morphism  $u_A : A_s \rightarrow A_{s'}$  in  $\widehat{\mathcal{D}}$  with  $h \cdot u_A = u_{A'} \cdot h$  for all  $h$ . It follows that the square (5) commutes, as it commutes by Remark E.7 when postcomposed with the limit projections  $h$ .

$$\begin{array}{ccc}
 (\hat{T}\Sigma)_s & \xrightarrow{\hat{u}} & (\hat{T}\Sigma)_{s'} \\
 \hat{e} \downarrow & & \downarrow \hat{e} \\
 A_s & \xrightarrow{u_A} & A_{s'} \\
 h \downarrow & & \downarrow h \\
 A'_s & \xrightarrow{u_{A'}} & A'_{s'}
 \end{array}$$

(ii) $\Rightarrow$ (i) Let  $\hat{e} : \hat{T}\Sigma \twoheadrightarrow A$  be an epimorphism in  $\widehat{\mathcal{D}}^S$  with the lifting property (5). We need to show that  $A$  has a  $\widehat{\mathbf{T}}$ -algebra structure such that  $A$  is profinite and  $\hat{e}$  is a  $\widehat{\mathbf{T}}$ -homomorphism.

(a) We first prove an auxiliary result. Let  $g : \hat{\mathbf{T}}\Sigma \twoheadrightarrow B$  be a surjective  $\widehat{\mathbf{T}}$ -homomorphism with  $B \in \mathbf{Alg}_f \widehat{\mathbf{T}}$ , and form the pushout  $p = g' \cdot \hat{e} = e' \cdot g : \hat{T}\Sigma \twoheadrightarrow P$  of  $\hat{e}$  and  $g$  in  $\widehat{\mathcal{D}}^S$ . We claim that  $P$  carries a  $\widehat{\mathbf{T}}$ -algebra structure making  $p$  a  $\widehat{\mathbf{T}}$ -homomorphism. To see this consider the diagram below:

$$\begin{array}{ccccc}
 & & (\hat{T}\Sigma)_{s'} & & \\
 & \nearrow \hat{e} & \uparrow \hat{u} & \nwarrow g & \\
 & (\hat{T}\Sigma)_s & & & \\
 & \nwarrow \hat{e} & \downarrow p & \nearrow g & \\
 A_{s'} & \xleftarrow{u_A} & A_s & & B_s \xrightarrow{u_B} B_{s'} \\
 & \nwarrow g' & \downarrow g' & \nearrow e' & \\
 & & P_s & & \\
 & \nwarrow g' & \downarrow u_P & \nearrow e' & \\
 & & P_{s'} & &
 \end{array}$$

By hypothesis there exists for each  $u : (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathcal{U}_\Sigma$  a morphism  $u_A$  making the upper left square commute. Likewise, since  $\mathcal{U}_\Sigma$  is a unary presentation, there exists by Remark E.7 a morphism  $u_B$  making the upper right square commute. Then the morphisms  $g' \cdot u_A$  and  $e' \cdot u_B$  form a compatible family, so by the universal property of the pushout there exists a unique morphism  $u_P : P_s \rightarrow P_{s'}$  in  $\widehat{\mathcal{D}}$  making the two lower squares commute. Thus the whole diagram above commutes, which shows that  $u_P \cdot p = p \cdot \hat{u}$  for all  $u : (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathcal{U}_\Sigma$ . Since  $\mathcal{U}_\Sigma$  forms a unary presentation and  $P$  is finite, it follows from Remark E.7 that  $P$  carries a  $\widehat{\mathbf{T}}$ -algebra structure making  $p$  a  $\widehat{\mathbf{T}}$ -homomorphism, as desired.

(b) Let  $\hat{U} : \mathbf{Alg}_f \widehat{\mathbf{T}} \rightarrow \widehat{\mathcal{D}}^S$  denote the forgetful functor, and let  $\mathcal{S}$  be the full subcategory of  $(A \downarrow \hat{U})$  on all surjective morphisms  $h : A \twoheadrightarrow \hat{U}(A', \alpha')$  for which  $h \cdot \hat{e} : \hat{\mathbf{T}}\Sigma \twoheadrightarrow (A', \alpha')$  is a  $\widehat{\mathbf{T}}$ -homomorphism. Let us first verify that the category  $\mathcal{S}$  is cofiltered, see A.7. First,  $\mathcal{S}$  is nonempty because the image of the unique morphism  $f : A \rightarrow 1$  into the terminal  $\widehat{\mathbf{T}}$ -algebra lies in  $\mathcal{S}$ . Second, any two  $h_i : A \twoheadrightarrow \hat{U}(A'_i, \alpha'_i)$  ( $i = 0, 1$ ) in  $\mathcal{S}$  have a common predecessor. To see this, form the product  $\pi_i : A'_0 \times A'_1 \rightarrow A'_i$  in  $\mathbf{Alg} \widehat{\mathbf{T}}$  and factorize

the morphism  $\langle h_0, h_1 \rangle : A \rightarrow A'_0 \times A'_1$  in  $\widehat{\mathcal{D}}^S$  as  $\langle h_0, h_1 \rangle = m \cdot h$  with  $h$  surjective and  $m$  injective.

$$\begin{array}{c}
 \hat{T}\Sigma \\
 \downarrow \hat{e} \\
 A \\
 \swarrow h_0 \quad \downarrow h \quad \searrow h_1 \\
 A'_0 \quad A' \quad A'_1 \\
 \swarrow \pi_0 \quad \downarrow m \quad \searrow \pi_1 \\
 A'_0 \times A'_1
 \end{array}$$

Since the projections  $\pi_i$  are jointly monomorphic and  $\pi_i \cdot m \cdot h \cdot \hat{e} = h_i \cdot \hat{e}$  is a  $\widehat{\mathbf{T}}$ -homomorphism, so is  $m \cdot h \cdot \hat{e}$ . Furthermore, since the factorization system of  $\widehat{\mathcal{D}}^S$  lifts to  $\mathbf{Alg} \widehat{\mathbf{T}}$ , see Remark C.3.7, there exists a  $\widehat{\mathbf{T}}$ -algebra structure  $(A', \alpha')$  on  $A'$  such that  $h \cdot \hat{e}$  and  $m$  are  $\widehat{\mathbf{T}}$ -homomorphisms. Thus  $h : A \rightarrow \hat{U}(A', \alpha')$  lies in  $\mathcal{S}$  and is the desired predecessor of  $h_0$  and  $h_1$ .

We claim that  $A$  is the cofiltered limit of the diagram

$$\mathcal{S} \xrightarrow{\pi} \widehat{\mathcal{D}}^S, \quad (h : A \rightarrow U(A', \alpha')) \rightarrow A' \quad (6)$$

with limit projections  $h : A \rightarrow A'$ . To this end verify the criterion of A.7, i.e. we show that any morphism  $f : A \rightarrow X$  with  $X \in \mathcal{D}_f^S$  factors through some  $h$ . The proof is illustrated by the diagram below:

$$\begin{array}{ccccc}
 \hat{T}\Sigma & \xrightarrow{g^+} & B & & \\
 \downarrow \hat{e} & \searrow p & \swarrow e' & & \downarrow s \\
 & & P & & \\
 \uparrow h & \swarrow & \searrow s' & & \downarrow \\
 A & \xrightarrow{f} & X & & 
 \end{array}$$

The morphism  $f \cdot \hat{e}$  factors through the cofiltered limit cone defining  $\hat{T}\Sigma$ , because  $X$  is finitely copresentable in  $\widehat{\mathcal{D}}^S$  (see Remark C.1). That is, there exists a surjective  $\mathbf{T}$ -homomorphism  $g : \mathbf{T}\Sigma \rightarrow B$  with  $B \in \mathbf{Alg}_f \mathbf{T}$  and a morphism  $s : B \rightarrow X$  in  $\widehat{\mathcal{D}}^S$  with  $s \cdot g^+ = f \cdot \hat{e}$ . Form the pushout  $p = h \cdot \hat{e} = e' \cdot g^+ : \hat{T}\Sigma \rightarrow P$  of  $\hat{e}$  and  $g^+$  in  $\widehat{\mathcal{D}}^S$ . Then the morphisms  $f$  and  $s$  form a compatible family, so the universal property of the pushout yields an  $s' : P \rightarrow X$  in  $\widehat{\mathcal{D}}^S$  with  $s' \cdot e' = s$  and  $s' \cdot h = f$ . Moreover, by part (a) the object  $P$  carries a  $\widehat{\mathbf{T}}$ -algebra structure  $(P, \varrho)$  making  $p$  a  $\widehat{\mathbf{T}}$ -homomorphism. Since  $p = h \cdot \hat{e}$ , this implies that  $h : A \rightarrow \hat{U}(P, \varrho)$  is an object in  $\mathcal{S}$ , so  $f = s' \cdot h$  is the desired factorization of  $f$ .

- (c) Since the forgetful functor from  $\mathbf{Alg} \widehat{\mathbf{T}}$  to  $\widehat{\mathcal{D}}^S$  creates limits, see A.3, it follows from (b) that there is a unique  $\widehat{\mathbf{T}}$ -algebra structure  $\alpha : \hat{T}A \rightarrow A$  on  $A$  making  $(h : (A, \alpha) \rightarrow (A', \alpha'))$  a cofiltered limit cone in  $\mathbf{Alg} \widehat{\mathbf{T}}$ . Thus  $(A, \alpha)$  is profinite. To see that  $\hat{e} : \widehat{\mathbf{T}}\Sigma \rightarrow$

$(A, \alpha)$  is a  $\widehat{\mathbf{T}}$ -homomorphism, consider the diagram below:

$$\begin{array}{ccc}
 \widehat{T}\widehat{T}\Sigma & \xrightarrow{\hat{\mu}_\Sigma} & \widehat{T}\Sigma \\
 \hat{T}\hat{e} \downarrow & & \downarrow \hat{e} \\
 \widehat{T}A & \xrightarrow{\alpha} & A \\
 \hat{T}h \downarrow & & \downarrow h \\
 \widehat{T}A' & \xrightarrow{\alpha'} & A'
 \end{array}$$

The lower square commutes for all  $h : A \rightarrow \hat{U}(A', \alpha')$  in  $\mathcal{S}$  by the definition of  $\alpha$ , and the outer part commutes because by the definition of  $\mathcal{S}$  the morphism  $h \cdot \hat{e}$  is a  $\widehat{\mathbf{T}}$ -homomorphism. Thus also the upper square commutes, as it commutes when composed with the limit projections  $h$  in  $\widehat{\mathcal{D}}^S$ .  $\blacktriangleleft$

► **Remark E.9.** To prove Proposition 4.3 we first explain how to translate a local pseudovariety into a  $\Sigma$ -generated profinite  $\widehat{\mathbf{T}}$ -algebra and vice versa.

1. To each local pseudovariety  $\mathcal{P}$  of  $\Sigma$ -generated  $\mathbf{T}$ -algebras we associate a  $\Sigma$ -generated profinite  $\widehat{\mathbf{T}}$ -algebra  $\varphi_\Sigma^\mathcal{P} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow P_\Sigma^\mathcal{P}$  as follows. Viewed as full subcategory of the comma category  $(\mathbf{T}\Sigma \downarrow \mathbf{Alg}_f \mathbf{T})$ , the category  $\mathcal{P}$  is cofiltered because  $\mathcal{P}$  is closed under subdirect products. Let  $P_\Sigma^\mathcal{P}$  be the cofiltered limit of the diagram

$$\mathcal{P} \rightarrow \mathbf{Alg} \widehat{\mathbf{T}}, \quad (e : \mathbf{T}\Sigma \twoheadrightarrow A) \mapsto A,$$

and denote the limit projections by  $e_\mathcal{P}^* : P_\Sigma^\mathcal{P} \twoheadrightarrow A$ . They are surjective by Lemma B.2. Thus  $P_\Sigma^\mathcal{P}$  is a profinite  $\widehat{\mathbf{T}}$ -algebra. Moreover, the  $\widehat{\mathbf{T}}$ -homomorphisms  $e^+ : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow A$  (where  $e$  ranges over all elements of  $\mathcal{P}$ ) form a compatible family over the above diagram, so there exists a unique  $\widehat{\mathbf{T}}$ -homomorphism  $\varphi_\Sigma^\mathcal{P} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow P_\Sigma^\mathcal{P}$  with  $e^+ = e_\mathcal{P}^* \cdot \varphi_\Sigma^\mathcal{P}$  for all  $e \in \mathcal{P}$ . Note that  $\varphi_\Sigma^\mathcal{P}$  is surjective by Lemma B.1. This yields the desired  $\Sigma$ -generated profinite  $\widehat{\mathbf{T}}$ -algebra  $\varphi_\Sigma^\mathcal{P} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow P_\Sigma^\mathcal{P}$ .

2. Conversely, given a  $\Sigma$ -generated profinite  $\widehat{\mathbf{T}}$ -algebra  $\varphi_\Sigma : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow P_\Sigma$ , define  $\mathcal{P}^{\varphi_\Sigma}$  to be the class of all finite  $\Sigma$ -generated  $\mathbf{T}$ -algebras of the form

$$e = ( \mathbf{T}\Sigma \xrightarrow{\iota_\Sigma} V\widehat{\mathbf{T}}\Sigma \xrightarrow{V\varphi_\Sigma} VP_\Sigma \xrightarrow{Ve'} A ),$$

where  $e' : P_\Sigma \twoheadrightarrow A$  is a surjective  $\widehat{\mathbf{T}}$ -homomorphism with  $A \in \mathbf{Alg}_f \widehat{\mathbf{T}}$ . Note that any such morphism  $e$  is indeed a surjective  $\mathbf{T}$ -homomorphism by Remark C.3.5 and C.3.6. Clearly  $\mathcal{P}^{\varphi_\Sigma}$  forms a local pseudovariety of  $\Sigma$ -generated  $\mathbf{T}$ -algebras.

► **Lemma E.10.** For any local pseudovariety  $\mathcal{P}$  of  $\Sigma$ -generated  $\mathbf{T}$ -algebras we have  $\mathcal{P} = \mathcal{P}^{(\varphi_\Sigma^\mathcal{P})}$ .

**Proof.**  $\mathcal{P} \subseteq \mathcal{P}^{(\varphi_\Sigma^\mathcal{P})}$ : Let  $(e : \mathbf{T}\Sigma \twoheadrightarrow A) \in \mathcal{P}$ . Then for the corresponding limit projection  $e_\mathcal{P}^* : P_\Sigma^\mathcal{P} \twoheadrightarrow A$  we have

$$e = ( \mathbf{T}\Sigma \xrightarrow{\iota_\Sigma} V\widehat{\mathbf{T}}\Sigma \xrightarrow{V\varphi_\Sigma^\mathcal{P}} VP_\Sigma^\mathcal{P} \xrightarrow{Ve_\mathcal{P}^*} A )$$

by the definition of  $\varphi_\Sigma^\mathcal{P}$  and Remark 2.11.2. Therefore  $e \in \mathcal{P}^{(\varphi_\Sigma^\mathcal{P})}$  by the definition of  $\mathcal{P}^{(\varphi_\Sigma^\mathcal{P})}$ .

$\mathcal{P}^{(\varphi_\Sigma^\mathcal{P})} \subseteq \mathcal{P}$ : Let  $(e : \mathbf{T}\Sigma \twoheadrightarrow A) \in \mathcal{P}^{(\varphi_\Sigma^\mathcal{P})}$ . Thus there exists a surjective  $\widehat{\mathbf{T}}$ -homomorphism  $e' : P_\Sigma^\mathcal{P} \twoheadrightarrow A$  with  $A \in \mathbf{Alg}_f \widehat{\mathbf{T}}$  and

$$e = ( \mathbf{T}\Sigma \xrightarrow{\iota_\Sigma} V\widehat{\mathbf{T}}\Sigma \xrightarrow{V\varphi_\Sigma^\mathcal{P}} VP_\Sigma^\mathcal{P} \xrightarrow{Ve'} A ).$$

Since  $A$  is finitely copresentable in  $\mathbf{Alg} \hat{\mathbf{T}}$ , see Remark C.3.4, the  $\hat{\mathbf{T}}$ -homomorphism  $e'$  factors through the limit cone defining  $P_\Sigma^\mathcal{P}$ ; that is, there exists an  $h : \mathbf{T}\Sigma \twoheadrightarrow A'$  in  $\mathcal{P}$  and a  $\hat{\mathbf{T}}$ -homomorphism  $s : A' \twoheadrightarrow A$  with  $e' = s \cdot h_\mathcal{P}^*$ . Since  $e'$  is surjective, so is  $s$ . Then the commutative diagram below shows that  $e$  is a quotient of  $h \in \mathcal{P}$ , and thus lies in  $\mathcal{P}$  because  $\mathcal{P}$  is closed under quotients.

$$\begin{array}{ccccccc}
 & & e & & & & \\
 & \nearrow & & \searrow & & & \\
 T\Sigma & \xrightarrow{\iota_\Sigma} & V\hat{T}\Sigma & \xrightarrow{V\varphi_\Sigma^\mathcal{P}} & VP_\Sigma^\mathcal{P} & \xrightarrow{Vh_\mathcal{P}^*} & A' \twoheadrightarrow A \\
 & \searrow & & \nearrow & & & \\
 & & h^+ & & & & \\
 & \nwarrow & & \swarrow & & & \\
 & & h & & & & 
 \end{array}$$

◀

► **Lemma E.11.** *For each  $\Sigma$ -generated profinite  $\hat{\mathbf{T}}$ -algebra  $\varphi_\Sigma : \hat{\mathbf{T}}\Sigma \twoheadrightarrow P_\Sigma$  we have an isomorphism  $\varphi_\Sigma \cong \varphi_\Sigma^{(\mathcal{P}\varphi_\Sigma)}$ .*

► **Remark E.12.** More precisely, the lemma states that  $\varphi_\Sigma$  and  $\varphi_\Sigma^{(\mathcal{P}\varphi_\Sigma)}$  are isomorphic quotients of  $\hat{\mathbf{T}}\Sigma$ , i.e. there exists an isomorphism  $j_\Sigma : P_\Sigma \xrightarrow{\cong} P_\Sigma^{(\mathcal{P}\varphi_\Sigma)}$  with  $\varphi_\Sigma^{(\mathcal{P}\varphi_\Sigma)} = j_\Sigma \cdot \varphi_\Sigma$ .

**Proof.** Let  $(P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  be the full subcategory of  $(P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  on all surjective  $\hat{\mathbf{T}}$ -homomorphisms  $e' : \hat{\mathbf{T}}\Sigma \twoheadrightarrow A$  with  $A \in \mathbf{Alg}_f \hat{\mathbf{T}}$ . Consider the functor

$$F : (P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}}) \rightarrow \mathcal{P}^{\varphi_\Sigma}$$

that maps  $e' : P_\Sigma \twoheadrightarrow A$  to the  $\Sigma$ -generated finite  $\mathbf{T}$ -algebra

$$F(e') = (T\Sigma \xrightarrow{\iota_\Sigma} V\hat{T}\Sigma \xrightarrow{V\varphi_\Sigma} VP_\Sigma \xrightarrow{Ve'} A).$$

and acts as identity on morphisms. Note that  $F(e') \in \mathcal{P}^{\varphi_\Sigma}$  by the definition of  $\mathcal{P}^{\varphi_\Sigma}$ , so  $F$  is well-defined. We claim that  $F$  is an isomorphism. Indeed,  $F$  is injective on objects because  $\varphi_\Sigma$  is surjective and  $\iota_\Sigma$  is dense. The surjectivity on objects is the definition of  $\mathcal{P}^{\varphi_\Sigma}$ . The bijectivity on morphisms is clear.

Next observe that  $F$  commutes with the projection functors  $\pi$  and  $\pi'$ :

$$\begin{array}{ccc}
 (P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}}) & \xrightarrow{F} & \mathcal{P}^{\varphi_\Sigma} \\
 \searrow \pi & & \swarrow \pi' \\
 & \mathbf{Alg} \hat{\mathbf{T}} & 
 \end{array}$$

The limit of  $\pi$  is  $P_\Sigma$  by Corollary E.4, and the limit of  $\pi'$  is  $P_\Sigma^{(\mathcal{P}\varphi_\Sigma)}$  by the definition of  $P_\Sigma^{(\mathcal{P}\varphi_\Sigma)}$ . Since  $F$  is an isomorphism (in particular, a final functor) and limits are unique up to isomorphism, there is an isomorphism  $j_\Sigma : P_\Sigma \xrightarrow{\cong} P_\Sigma^{(\mathcal{P}\varphi_\Sigma)}$  with  $e' = F(e')_{\mathcal{P}\varphi_\Sigma}^* \cdot j_\Sigma$  for all  $e' : P_\Sigma \twoheadrightarrow A$  in  $(P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$ . Thus in the diagram below the outward triangle and all parts

except for the central triangle commute:

$$\begin{array}{c}
 \hat{T}\Sigma \\
 \downarrow \iota_\Sigma \\
 V\hat{T}\Sigma \\
 \begin{array}{ccc}
 \downarrow V\varphi_\Sigma & & \downarrow V\varphi_\Sigma^{(\mathcal{P}\varphi_\Sigma)} \\
 VP_\Sigma & \xrightarrow{Vj_\Sigma} & VP_\Sigma^{(\mathcal{P}\varphi_\Sigma)} \\
 \downarrow Ve' & & \downarrow VF(e')_{\mathcal{P}\varphi_\Sigma}^* \\
 A & & A
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow F(e') \\
 \searrow VF(e')^+
 \end{array}$$

It follows that the central triangle also commutes, as it commutes when precomposed with the dense map  $\iota_\Sigma$  and postcomposed with the limit projections  $VF(e')_{\mathcal{P}\varphi_\Sigma}^*$ . ◀

**Proof of Proposition 4.3.** By Lemma E.10 and E.11 the maps  $\mathcal{P} \mapsto \varphi_\Sigma^\mathcal{P}$  and  $\varphi_\Sigma \mapsto \mathcal{P}^{\varphi_\Sigma}$  are mutually inverse and thus give a bijection between the two posets. It only remains to prove that both maps are order-preserving. Given local pseudovarieties  $\varphi_\Sigma \leq \varphi'_\Sigma$ , we clearly have  $\mathcal{P}^{\varphi_\Sigma} \subseteq \mathcal{P}^{\varphi'_\Sigma}$  because every quotient of  $P_\Sigma$  is also a quotient of  $P'_\Sigma$ . Given local pseudovarieties  $\mathcal{P} \subseteq \mathcal{P}'$ , the morphisms  $e_{\mathcal{P}'}^* : P_{\Sigma}^{\mathcal{P}'} \twoheadrightarrow A$ , where  $e$  ranges over all  $e : \mathbf{T}\Sigma \twoheadrightarrow A$  in  $\mathcal{P}$ , form a compatible family over the diagram defining  $P_\Sigma^\mathcal{P}$ . Hence there exists a unique morphism  $q : P_\Sigma^\mathcal{P} \rightarrow P_\Sigma^{\mathcal{P}'}$  with  $e_{\mathcal{P}'}^* = e_\mathcal{P}^* \cdot q$  for all  $e \in \mathcal{P}$ . It follows that  $q \cdot \varphi_\Sigma^{\mathcal{P}'} = \varphi_\Sigma^\mathcal{P}$ , because this holds when postcomposed with the limit projections  $e_\mathcal{P}^*$ . Therefore  $\varphi_\Sigma^\mathcal{P} \leq \varphi_\Sigma^{\mathcal{P}'}$ .

$$\begin{array}{ccc}
 \hat{T}\Sigma & & \\
 \downarrow \varphi_{\Sigma^{\mathcal{P}'}}^\mathcal{P} & \searrow \varphi_\Sigma^\mathcal{P} & \\
 P_\Sigma^{\mathcal{P}'} & \xrightarrow{q} & P_\Sigma^\mathcal{P} \\
 \downarrow e_{\mathcal{P}'}^* & \nearrow e_\mathcal{P}^* & \\
 A & & A
 \end{array}$$

► **Remark E.13.** The *homomorphism theorem* states that given  $e : A \twoheadrightarrow B$  and  $f : A \twoheadrightarrow C$  in  $\widehat{\mathcal{D}}^S$  with  $e$  surjective, there exists a morphism  $g$  in  $\widehat{\mathcal{D}}^S$  with  $g \cdot e = f$  iff, for all sorts  $s$  and  $a, a' \in |A|_s$ ,  $e(a) = e(a')$  implies  $f(a) = f(a')$ . Indeed, there clearly is a  $\mathcal{D}^S$ -morphism  $g$  with this property, and it is continuous because  $A, B, C$  are compact Hausdorff spaces. Moreover, if  $A, B, C$  are  $\widehat{\mathbf{T}}$ -algebras and  $e$  and  $f$  are  $\widehat{\mathbf{T}}$ -homomorphisms, so is  $g$ . This follows from the fact that  $\hat{T}$  preserves epimorphisms, see Remark C.3.7.

**Details for Remark 4.4.** For a set  $E$  of profinite equations over  $\Sigma$ , let  $\mathcal{P}[E]$  denote the class of all  $\Sigma$ -generated  $\mathbf{T}$ -algebras satisfying all equations in  $E$ . Conversely, for a class  $\mathcal{P}$  of  $\Sigma$ -generated finite  $\mathbf{T}$ -algebras, let  $E[\mathcal{P}]$  be set of all profinite equations over  $\Sigma$  satisfied by all algebras in  $\mathcal{P}$ . The claim is that  $\mathcal{P}$  forms a local pseudovariety iff  $\mathcal{P} = \mathcal{P}[E]$  for some  $E$ .

The “if” direction is a straightforward verification. For the “only if” direction, suppose that  $\mathcal{P}$  is a local pseudovariety of  $\Sigma$ -generated  $\mathbf{T}$ -algebras, and let  $\varphi_\Sigma^\mathcal{P} : \hat{\mathbf{T}}\Sigma \twoheadrightarrow P_\Sigma^\mathcal{P}$  be the corresponding  $\Sigma$ -generated profinite  $\widehat{\mathbf{T}}$ -algebra, see Remark E.9.1. From the definition of  $\varphi_\Sigma^\mathcal{P}$  it immediately follows that  $\mathcal{P}$  satisfies a profinite equation  $u = v$  iff  $u$  and  $v$  are merged by



$\varphi_\Sigma^\mathcal{P}$ . We claim that  $\mathcal{P} = \mathcal{P}[E[\mathcal{P}]]$ . The inclusion  $\subseteq$  is trivial. To prove  $\supseteq$ , let  $e : \mathbf{T}\Sigma \rightarrow A$  be an element of  $\mathcal{P}[E[\mathcal{P}]]$ , i.e.  $e$  satisfies every equation that  $\mathcal{P}$  satisfies. By the homomorphism theorem, see Remark E.13, there exists a (surjective)  $\widehat{\mathbf{T}}$ -homomorphism  $h : P_\Sigma^\mathcal{P} \rightarrow A$  with  $e^+ = h \cdot \varphi_\Sigma^\mathcal{P}$ . Indeed, every pair  $u, v$  that is merged by  $\varphi_\Sigma^\mathcal{P}$  is a profinite equation  $u = v$  satisfied by  $\mathcal{P}$ . Thus  $u = v$  is satisfied by  $e$ , i.e.  $e^+$  merges  $u$  and  $v$ . Since  $A$  is finitely copresentable in  $\mathbf{Alg} \widehat{\mathbf{T}}$ , see Remark C.3.4,  $h$  factors through the limit cone defining  $P_\Sigma^\mathcal{P}$ ; that is, there exists an  $\bar{e} : \mathbf{T}\Sigma \rightarrow \bar{A}$  in  $\mathcal{P}$  and a  $\mathbf{T}$ -homomorphism  $g : \bar{A} \rightarrow A$  with  $h = g \cdot \bar{e}_\Sigma^*$ . Since  $h$  is surjective, so is  $g$ . Thus  $e$  is a quotient of  $e'$  (via  $g$ ) and hence lies in  $\mathcal{P}$ .  $\blacktriangleleft$

**Details for Example 4.7.2.** Let  $\mathbb{A} = \{(\Sigma, \emptyset) : \Sigma \in \mathbf{Set}_f\}$ . We prove that a finite  $\omega$ -semigroup  $A = (A_+, A_\omega)$  is  $\mathbb{A}$ -generated iff it is *complete*, i.e. every element  $a \in A_\omega$  can be expressed as an infinite product  $a = \pi(a_0, a_1, \dots)$  for some  $a_i \in A_+$ . For the “only if” direction, suppose that  $A$  is  $\mathbb{A}$ -generated, i.e. there exists a surjective  $\omega$ -semigroup morphism  $e : (\Sigma^+, \Sigma^\omega) \rightarrow (A_+, A_\omega)$  for some  $\Sigma \in \mathbf{Set}_f$ . For each  $a \in A_\omega$ , choose  $s_0 s_1 \dots \in \Sigma^\omega$  with  $a = e(s_0 s_1 \dots)$ . Then

$$a = e(s_0 s_1 \dots) = e(\pi(s_0, s_1, \dots)) = \pi(e(s_0), e(s_1), \dots),$$

which shows that  $A$  is complete. For the “if” direction, suppose that  $A$  is complete. Let  $\Sigma := A_+ \in \mathbf{Set}_f$ , and extend the map  $(id, \emptyset) : (\Sigma, \emptyset) \rightarrow (A_+, A_\omega)$  to an  $\omega$ -semigroup morphism  $e : (\Sigma^+, \Sigma^\omega) \rightarrow (A_+, A_\omega)$ , using that  $(\Sigma^+, \Sigma^\omega)$  is the free  $\omega$ -semigroup on  $(\Sigma, \emptyset)$ . Clearly the component  $e : \Sigma^+ \rightarrow A_+$  is surjective because  $e(a) = a$  for all  $a \in A_+$ . To show that also the component  $e : \Sigma^\omega \rightarrow A_\omega$  is surjective, let  $a \in A_\omega$  and choose elements  $a_i \in A_+$  with  $a = \pi(a_0, a_1, \dots)$ , using the completeness of  $A$ . It follows that

$$a = \pi(a_0, a_1, \dots) = \pi(e(a_0), e(a_1), \dots) = e(\pi(a_0, a_1, \dots)).$$

Thus  $e$  is surjective, which proves that  $A$  is  $\mathbb{A}$ -generated.  $\blacktriangleleft$

**Details for Remark 4.8.** See Lemma C.4.  $\blacktriangleleft$

The proof of Proposition 4.10, establishing the equivalence of profinite theories and pseudovarieties, is achieved through a sequence of lemmas. First an auxiliary result:

► **Lemma E.14.** *Given  $\Sigma, \Delta \in \mathbf{Set}_f^S$ , a surjective  $\widehat{\mathbf{T}}$ -homomorphism  $e : \widehat{\mathbf{T}}\Sigma \rightarrow A$  with  $A \in \mathbf{Alg}_f \widehat{\mathbf{T}}$  and a  $\widehat{\mathbf{T}}$ -homomorphism  $h : \widehat{\mathbf{T}}\Delta \rightarrow A$ , there exists a  $\mathbf{T}$ -homomorphism  $g : \mathbf{T}\Delta \rightarrow \mathbf{T}\Sigma$  with  $h = e \cdot \hat{g}$ .*

**Proof.** Let  $e' = Ve \cdot \iota_\Sigma : \mathbf{T}\Delta \rightarrow A$  and  $h' = Vh \cdot \iota_\Sigma : \mathbf{T}\Sigma \rightarrow A$  be the restrictions of  $e$  and  $h$  to  $\mathbf{T}$ -homomorphisms, see Remark C.3.5. Since the free object  $\Delta$  is projective in the  $S$ -sorted variety  $\mathcal{D}^S$ , and  $e'$  is surjective by Remark C.3.6, there exists a morphism  $g' : \Delta \rightarrow \mathbf{T}\Sigma$  in  $\mathcal{D}^S$  with  $h' \cdot \eta_\Delta = e' \cdot g'$ . Since  $\mathbf{T}\Delta$  is the free  $\mathbf{T}$ -algebra on  $\Delta$ , see A.2, we can extend  $g'$  to a  $\mathbf{T}$ -homomorphism  $g : \mathbf{T}\Delta \rightarrow \mathbf{T}\Sigma$ . Then  $\hat{g}$  has the desired property: the lower triangle in the diagram below commutes, because it commutes when precomposed with the dense map  $\iota_\Delta$

and the unit  $\eta_\Delta$ .

$$\begin{array}{ccc}
 \Delta & & \\
 \eta_\Delta \downarrow & \searrow g' & \\
 T\Delta & \xrightarrow{g} & T\Sigma \\
 \iota_\Delta \downarrow & & \downarrow \iota_\Sigma \\
 VT\Delta & \xrightarrow{V\hat{g}} & VT\Sigma \xrightarrow{Ve} A \\
 & \searrow Vh & \nearrow \\
 & & 
 \end{array}$$

◀

► **Lemma E.15.** *Let  $\varphi$  be a profinite theory, and  $(A, \alpha)$  be an  $\mathbb{A}$ -generated finite  $\mathbf{T}$ -algebra. The following statements are equivalent:*

- (i) *There exists a surjective  $\hat{\mathbf{T}}$ -homomorphism  $e : P_\Sigma \twoheadrightarrow (A, \alpha^+)$  for some  $\Sigma \in \mathbb{A}$ .*
- (ii) *Every  $\hat{\mathbf{T}}$ -homomorphism  $h : \hat{\mathbf{T}}\Delta \rightarrow (A, \alpha^+)$  with  $\Delta \in \mathbb{A}$  factors through  $\varphi_\Delta$ :*

$$\begin{array}{ccc}
 \hat{\mathbf{T}}\Delta & & \\
 \varphi_\Delta \downarrow & \searrow h & \\
 P_\Delta & \xrightarrow{h'} & A
 \end{array}$$

**Proof.** (i)⇒(ii) Given a  $\hat{\mathbf{T}}$ -homomorphism  $h : \hat{\mathbf{T}}\Delta \rightarrow A$ , choose a  $\mathbf{T}$ -homomorphism  $g : \mathbf{T}\Delta \rightarrow \mathbf{T}\Sigma$  with  $h = e \cdot \varphi_\Sigma \cdot \hat{g}$ , see Lemma E.14. Since  $\varphi$  is a profinite theory, there exists a  $\hat{\mathbf{T}}$ -homomorphism  $g_P : P_\Delta \rightarrow P_\Sigma$  with  $g_P \cdot \varphi_\Delta = \varphi_\Sigma \cdot \hat{g}$ . Thus  $h = (e \cdot g_P) \cdot \varphi_\Delta$  is the desired factorization of  $h$  through  $\varphi_\Delta$ .

$$\begin{array}{ccc}
 \hat{\mathbf{T}}\Delta & \xrightarrow{\hat{g}} & \hat{\mathbf{T}}\Sigma \\
 \varphi_\Delta \downarrow & & \downarrow \varphi_\Sigma \\
 P_\Delta & \xrightarrow{g_P} & P_\Sigma \xrightarrow{e} A \\
 & \searrow h & \nearrow
 \end{array}$$

(ii)⇒(i) Since  $A$  is  $\mathbb{A}$ -generated, there exists a surjective  $\mathbf{T}$ -homomorphism  $e : \mathbf{T}\Sigma \twoheadrightarrow A$  for some  $\Sigma \in \mathbb{A}$ . By hypothesis the surjective  $\hat{\mathbf{T}}$ -homomorphism  $e^+ : \hat{\mathbf{T}}\Sigma \twoheadrightarrow A$  factors through  $\varphi_\Sigma$ . Thus  $A$  is a quotient of  $P_\Sigma$ . ◀

► **Lemma E.16.** *Let  $\varphi$  be a profinite theory. Then the class  $\mathcal{V}_\varphi$  of all  $\mathbb{A}$ -generated finite  $\mathbf{T}$ -algebras  $(A, \alpha)$  satisfying the equivalent properties of Lemma E.15 forms a pseudovariety.*

**Proof.** Let  $A_i$  ( $i \in I$ ) be finitely many objects in  $\mathcal{V}_\varphi$ . Form the product  $\pi_i : \prod_i A_i \rightarrow A_i$  and let  $m : A \hookrightarrow \prod_i A_i$  be an  $\mathbb{A}$ -generated subalgebra. We show that any  $\hat{\mathbf{T}}$ -homomorphism  $h : \hat{\mathbf{T}}\Delta \rightarrow A$  factors through  $\varphi_\Delta$ . For each  $i$ , there exists a  $\hat{\mathbf{T}}$ -homomorphism  $h_i : P_\Delta \rightarrow A_i$  with  $h_i \cdot \varphi_\Delta = \pi_i \cdot m \cdot h$ , because  $A_i \in \mathcal{V}_\varphi$ . Then  $h$  factors through  $\varphi_\Delta$  via the diagonal fill-in property, which shows that  $A \in \mathcal{V}_\varphi$ .

$$\begin{array}{ccccc}
 \mathbf{T}\Delta & \xrightarrow{\varphi_\Delta} & P_\Delta & & \\
 h \downarrow & \swarrow & \downarrow \langle h_i \rangle & \searrow h_i & \\
 A & \xrightarrow{m} & \prod_i A_i & \xrightarrow{\pi_i} & A_i
 \end{array}$$

The closure of  $\mathcal{V}_\varphi$  under quotients follows from Lemma E.15(i).  $\blacktriangleleft$

The reverse passage from pseudovarieties to profinite theories requires some preparation.

► **Lemma E.17.** *Let  $\mathcal{V}$  be a class of  $\mathbb{A}$ -generated finite  $\mathbf{T}$ -algebras closed under  $\mathbb{A}$ -generated subalgebras of finite products. Then for each  $\Sigma \in \mathbb{A}$  the comma categories  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  and  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  of all (surjective)  $\mathbf{T}$ -homomorphisms  $h : \mathbf{T}\Sigma \rightarrow A$  with  $A \in \mathcal{V}$  are cofiltered.*

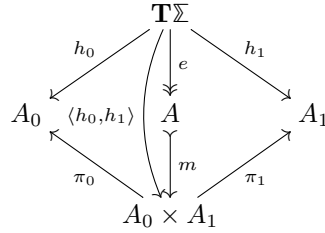
**Proof.** We only show that  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  is cofiltered; the argument for  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  is analogous. To this end we verify the criterion of A.7.

- (i)  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  is nonempty: let  $h : \mathbf{T}\Sigma \rightarrow 1$  be the unique  $\mathbf{T}$ -homomorphism into the terminal  $\mathbf{T}$ -algebra, and consider its factorization

$$h = (\mathbf{T}\Sigma \xrightarrow{e} A \xrightarrow{m} 1)$$

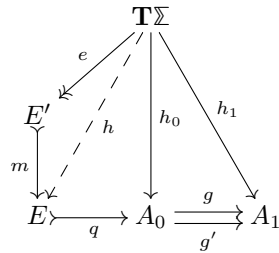
Then  $A$  is an  $\mathbb{A}$ -generated subalgebra of  $1$  (the empty product) and thus lies in  $\mathcal{V}$ . Hence  $e \in (\mathbf{T}\Sigma \downarrow \mathcal{V})$ .

- (ii) Given  $h_i : \mathbf{T}\Sigma \rightarrow A_i$  ( $i = 0, 1$ ) in  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$ , form the product  $\pi_i : A_0 \times A_1 \rightarrow A_i$  in  $\mathbf{Alg} \mathbf{T}$  and factorize the  $\mathbf{T}$ -homomorphism  $\langle h_0, h_1 \rangle : \mathbf{T}\Sigma \rightarrow A_0 \times A_1$  as  $\langle h_0, h_1 \rangle = m \cdot e$  with  $e$  surjective and  $m$  injective.



Then  $A \in \mathcal{V}$ , being an  $\mathbb{A}$ -generated subalgebra of the product  $A_0 \times A_1$ , and we have the morphisms  $\pi_i \cdot m : e \rightarrow h_i$  in  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$ .

- (iii) Given  $h_i : \mathbf{T}\Sigma \rightarrow A_i$  ( $i = 0, 1$ ) in  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  and two morphisms  $g, g' : h_0 \rightarrow h_1$ , form the equalizer  $q : E \rightarrow A_0$  of  $g$  and  $g'$  in  $\mathbf{Alg} \mathbf{T}$ . Since  $g \cdot h_0 = g' \cdot h_0$ , the universal property of  $q$  gives a unique  $\mathbf{T}$ -homomorphism  $h : \mathbf{T}\Sigma \rightarrow E$  with  $h_0 = q \cdot h$ . Let  $h = e \cdot m$  be the (surjective, injective) factorization of  $h$ , see the diagram below:



Then  $E'$  lies in  $\mathcal{V}$ , being an  $\mathbb{A}$ -generated subalgebra of  $A_0 \in \mathcal{V}$ . It follows that  $q \cdot m : e \rightarrow h_0$  is a morphism in  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  merging  $g$  and  $g'$ , i.e.  $g \cdot (q \cdot m) = g' \cdot (q \cdot m)$ .  $\blacktriangleleft$

► **Remark E.18.** We review a construction given in [13]. As in the above lemma, let  $\mathcal{V}$  be a class of  $\mathbb{A}$ -generated finite  $\mathbf{T}$ -algebras closed under  $\mathbb{A}$ -generated subalgebras of finite products.

1. In analogy the profinite monad  $\widehat{\mathbf{T}}$ , see Theorem 2.9 and Remark C.3.1/2, one can construct the *pro- $\mathcal{V}$  monad*  $\widehat{\mathbf{T}}_{\mathcal{V}} = (\widehat{T}_{\mathcal{V}}, \widehat{\eta}^{\mathcal{V}}, \widehat{\mu}^{\mathcal{V}})$  of  $\mathbf{T}$ . This is the codensity monad of the forgetful functor

$$\mathcal{V} \mapsto \mathbf{Alg}_f \mathbf{T} \rightarrow \mathcal{D}_f^S \xrightarrow{\cong} \widehat{\mathcal{D}}_f^S \mapsto \widehat{\mathcal{D}}^S.$$

By the limit formula for right Kan extensions, the object  $\widehat{T}_{\mathcal{V}}D$  for  $D \in \mathcal{D}_f^S$  is the limit of the diagram

$$(\mathbf{T}D \downarrow \mathcal{V}) \rightarrow \widehat{\mathcal{D}}^S, \quad (h : \mathbf{T}D \rightarrow (A, \alpha)) \mapsto A.$$

We denote the limit projections by

$$h_{\mathcal{V}}^+ : \widehat{T}_{\mathcal{V}}D \rightarrow A. \quad (7)$$

If  $D = \Sigma$  with  $\Sigma \in \mathbb{A}$ , the above limit is cofiltered by Lemma E.17, and one can restrict to the cofiltered subdiagram  $(\mathbf{T}\Sigma \downarrow \mathcal{V}) \rightarrow \widehat{\mathcal{D}}^S$  (cf. Remark 2.11.1).

2. For each  $(A, \alpha) \in \mathcal{V}$  we have the  $\mathbf{T}$ -homomorphism  $\alpha : \mathbf{T}A \rightarrow (A, \alpha)$ , and thus the limit projection  $\alpha_{\mathcal{V}}^+ : \widehat{T}_{\mathcal{V}}A \rightarrow A$ . Then the following squares commute for all  $\mathbf{T}$ -homomorphisms  $h : \mathbf{T}D \rightarrow (A, \alpha)$ :

$$\begin{array}{ccc} D & \xrightarrow{\widehat{\eta}_D^{\mathcal{V}}} & \widehat{T}_{\mathcal{V}}D \\ & \searrow h\eta_D & \downarrow h_{\mathcal{V}}^+ \\ & & A \end{array} \quad \begin{array}{ccc} \widehat{T}_{\mathcal{V}}\widehat{T}_{\mathcal{V}}D & \xrightarrow{\widehat{\mu}_D^{\mathcal{V}}} & \widehat{T}_{\mathcal{V}}D \\ \widehat{T}_{\mathcal{V}}h_{\mathcal{V}}^+ \downarrow & & \downarrow h_{\mathcal{V}}^+ \\ \widehat{T}_{\mathcal{V}}A & \xrightarrow{\alpha_{\mathcal{V}}^+} & A \end{array} \quad (8)$$

3. The universal property of right Kan extensions gives a monad morphism  $\varphi^{\mathcal{V}} : \widehat{\mathbf{T}} \rightarrow \widehat{\mathbf{T}}_{\mathcal{V}}$  (see A.1). For  $D \in \mathcal{D}_f^S$  the morphisms  $h^+$  form a compatible family over the diagram defining  $\widehat{T}_{\mathcal{V}}D$ , and the component  $\varphi_D^{\mathcal{V}}$  is the unique morphism in  $\widehat{\mathcal{D}}^S$  making the triangle below commute for all  $\mathbf{T}$ -homomorphisms  $h : \mathbf{T}D \rightarrow A$  with  $A \in \mathcal{V}$ :

$$\begin{array}{ccc} \widehat{T}D & \xrightarrow{\varphi_D^{\mathcal{V}}} & \widehat{T}_{\mathcal{V}}D \\ h^+ \downarrow & \swarrow h_{\mathcal{V}}^+ & \\ A & & \end{array} \quad (9)$$

By Remark E.18.1 and Lemma B.1, each component  $\varphi_{\Sigma}^{\mathcal{V}}$  with  $\Sigma \in \mathbb{A}$  is surjective. Note further that, since  $\varphi^{\mathcal{V}}$  is monad morphism,  $\varphi_{\Sigma}^{\mathcal{V}}$  is a  $\widehat{\mathbf{T}}$ -homomorphism

$$\varphi_{\Sigma}^{\mathcal{V}} : \widehat{\mathbf{T}}\Sigma \rightarrow (\widehat{T}_{\mathcal{V}}\Sigma, \widehat{\mu}_{\Sigma}^{\mathcal{V}} \cdot \varphi_{\widehat{T}_{\mathcal{V}}\Sigma}^{\mathcal{V}}).$$

► **Lemma E.19.** *Let  $\mathcal{V}$  be a pseudovariety of  $\mathbf{T}$ -algebras. Then the family*

$$\varphi_{\mathcal{V}} = (\varphi_{\Sigma}^{\mathcal{V}} : \widehat{\mathbf{T}}\Sigma \rightarrow \widehat{T}_{\mathcal{V}}\Sigma)_{\Sigma \in \mathbb{A}}$$

*forms a profinite theory.*

**Proof.** (1) For all  $\mathbf{T}$ -homomorphisms  $h : \mathbf{T}\Sigma \rightarrow (A, \alpha)$  with  $\Sigma \in \mathbb{A}$  and  $(A, \alpha) \in \mathcal{V}$  we have

the following commutative diagram.

$$\begin{array}{ccccc}
 \hat{T}\hat{T}_V\mathbb{Z} & \xrightarrow{\varphi_{\hat{T}_V\mathbb{Z}}^\vee} & \hat{T}_V\hat{T}_V\mathbb{Z} & \xrightarrow{\hat{\mu}_\mathbb{Z}^\vee} & \hat{T}_V\mathbb{Z} \\
 \hat{T}h_V^+ \downarrow & & \hat{T}_V h_V^+ \downarrow & & \downarrow h_V^+ \\
 \hat{T}A & \xrightarrow{\varphi_A^\vee} & \hat{T}_V A & \xrightarrow{\alpha_V^+} & A \\
 & \searrow \alpha^+ & & & \nearrow
 \end{array}$$

Indeed, the right square commutes by Remark E.18.2, the left square commutes by the naturality of  $\varphi^\vee$ , and for the lower triangle see Remark E.18.3. Since the forgetful functor from  $\mathbf{Alg} \hat{\mathbf{T}}$  to  $\hat{\mathcal{D}}^S$  reflects limits, see A.3, this shows that the  $\hat{\mathbf{T}}$ -homomorphisms

$$h_V^+ : (\hat{T}_V\mathbb{Z}, \hat{\mu}_\mathbb{Z}^\vee \cdot \varphi_{\hat{T}_V\mathbb{Z}}^\vee) \rightarrow (A, \alpha^+),$$

form a cofiltered limit cone in  $\mathbf{Alg} \hat{\mathbf{T}}$ . Hence the  $\hat{\mathbf{T}}$ -algebra  $(\hat{T}_V\mathbb{Z}, \hat{\mu}_\mathbb{Z}^\vee \cdot \varphi_{\hat{T}_V\mathbb{Z}}^\vee)$  is profinite.

(2) Given a  $\mathbf{T}$ -homomorphism  $g : \mathbf{T}\Delta \rightarrow \mathbf{T}\mathbb{Z}$  with  $\Sigma, \Delta \in \mathbb{A}$ , the morphisms  $(hg)_V^+$  (where  $h$  ranges over all  $\mathbf{T}$ -homomorphisms  $h : \mathbf{T}\mathbb{Z} \rightarrow A$  with  $A \in \mathcal{V}$ ) form a compatible family over the diagram defining  $\hat{T}_V\mathbb{Z}$ . Thus there exists a unique  $g' : \hat{T}_V\Delta \rightarrow \hat{T}_V\mathbb{Z}$  with  $(hg)_V^+ = h_V^+ \cdot g'$  for all  $h$ . It follows that the upper square in the following diagram commutes, as it commutes when composed with the limit projections  $h_V^+$  (the outside commutes due to Lemma C.4).

$$\begin{array}{ccc}
 \hat{\mathbf{T}}\Delta & \xrightarrow{\hat{g}} & \hat{\mathbf{T}}\mathbb{Z} \\
 \varphi_\Delta^\vee \downarrow & & \downarrow \varphi_\mathbb{Z}^\vee \\
 \hat{T}_V\Delta & \xrightarrow{g'} & \hat{T}_V\mathbb{Z} \\
 (hg)_V^+ \searrow & & \swarrow h_V^+ \\
 & A &
 \end{array}$$

The following two lemmas demonstrate that the constructions  $\varphi \mapsto \mathcal{V}_\varphi$  and  $\mathcal{V} \mapsto \varphi_\mathcal{V}$  of Lemma E.16 and E.19 are mutually inverse.

► **Lemma E.20.** *For any pseudovariety  $\mathcal{V}$  of  $\mathbf{T}$ -algebras we have  $\mathcal{V} = \mathcal{V}_{(\varphi_\mathcal{V})}$ .*

**Proof.**  $\mathcal{V} \subseteq \mathcal{V}_{(\varphi_\mathcal{V})}$ : Let  $A \in \mathcal{V}$ . Since  $A$  is  $\mathbb{A}$ -generated, there exists a surjective  $\mathbf{T}$ -homomorphism  $e : \mathbf{T}\mathbb{Z} \twoheadrightarrow A$  with  $\Sigma \in \mathbb{A}$ . Then we have the surjective  $\hat{\mathbf{T}}$ -homomorphism  $e_V^+ : \hat{T}_V\mathbb{Z} \twoheadrightarrow A$  and therefore  $A \in \mathcal{V}_{(\varphi_\mathcal{V})}$ .

$\mathcal{V}_{(\varphi_\mathcal{V})} \subseteq \mathcal{V}$ : Let  $A \in \mathcal{V}_{(\varphi_\mathcal{V})}$ . Since  $A$  is  $\mathbb{A}$ -generated, there exists a surjective  $\mathbf{T}$ -homomorphism  $e : \mathbf{T}\mathbb{Z} \twoheadrightarrow A$  with  $\Sigma \in \mathbb{A}$ . Thus we have the surjective  $\hat{\mathbf{T}}$ -homomorphism  $e^+ : \hat{\mathbf{T}}\mathbb{Z} \twoheadrightarrow A$ . By the definition of  $\mathcal{V}_{(\varphi_\mathcal{V})}$  there exists a (surjective)  $\hat{\mathbf{T}}$ -homomorphism  $e' : \hat{T}_V\mathbb{Z} \twoheadrightarrow A$  with  $e^+ = e' \cdot \varphi_\Sigma^\vee$ . Since the finite  $\hat{\mathbf{T}}$ -algebra  $A$  is finitely copresentable in  $\mathbf{Alg} \hat{\mathbf{T}}$ , see Remark 2.11.4, the homomorphism  $e'$  factors through the limit cone defining  $\hat{T}_V\mathbb{Z}$ ; that is, there exist  $\mathbf{T}$ -homomorphisms  $h : \mathbf{T}\mathbb{Z} \rightarrow B$  and  $e'' : B \rightarrow A$  with  $B \in \mathcal{V}$  and  $e' = e'' \cdot h_V^+$ . Since  $e'$  is surjective, so is  $e''$ . Hence the closure of  $\mathcal{V}$  under quotients implies

that  $A \in \mathcal{V}$ .

$$\begin{array}{ccc}
 \hat{T}\Sigma & \xrightarrow{\varphi_\Sigma^\mathcal{V}} & \hat{T}_\mathcal{V}\Sigma \\
 \downarrow e^+ & \searrow e' & \downarrow h_\mathcal{V}^+ \\
 A & \xleftarrow{e''} & B
 \end{array}$$

◀

► **Lemma E.21.** For any profinite theory  $\varphi = (\varphi_\Sigma : \hat{\mathbf{T}}\Sigma \twoheadrightarrow P_\Sigma)$  we have  $\varphi \cong \varphi_{(\mathcal{V}_\varphi)}$ .

► **Remark E.22.** More precisely, letting  $\mathcal{V} := \mathcal{V}_\varphi$ , the lemma states that there is an isomorphism  $j_\Sigma : \hat{T}_\mathcal{V}\Sigma \xrightarrow{\cong} P_\Sigma$  with  $\varphi_\Sigma = j_\Sigma \cdot \varphi_\Sigma^\mathcal{V}$  for each  $\Sigma \in \mathbf{Set}_f$ .

**Proof.** Let  $\mathcal{V} := \mathcal{V}_\varphi$ . Every surjective  $\hat{\mathbf{T}}$ -homomorphism  $e : P_\Sigma \twoheadrightarrow A$  with  $A \in \mathbf{Alg}_f \hat{\mathbf{T}}$  yields the surjective  $\mathbf{T}$ -homomorphism

$$e' = (T\Sigma \xrightarrow{\iota_\Sigma} V\hat{T}\Sigma \xrightarrow{V\varphi_\Sigma} VP_\Sigma \xrightarrow{Ve} A),$$

see Remark 2.11.5. Moreover,  $A \in \mathcal{V}$  by the definition of  $\mathcal{V}$ . Thus the map  $e \mapsto e'$  defines a functor (acting as identity on morphisms)

$$F : (P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}}) \rightarrow (\mathbf{T}\Sigma \downarrow \mathcal{V}),$$

where  $(P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  and  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  are the full subcategories of the comma categories  $(P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  and  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  on surjective homomorphisms. We claim that  $F$  is final, see A.8. Since  $(P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  is cofiltered, this requires to show that (i) for any object  $e'$  in  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  there exists a morphism  $F(e) \rightarrow e'$  in  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$  for some  $e \in (P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$ , and (ii) any two parallel morphisms  $F(e) \rightrightarrows e'$  are merged by some morphism in  $(P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$ . For (i), let  $e' : \mathbf{T}\Sigma \twoheadrightarrow A$  be an object of  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$ . Then we have the surjective  $\hat{\mathbf{T}}$ -homomorphism  $(e')^+ : \hat{\mathbf{T}}\Sigma \twoheadrightarrow A$ . Since  $A \in \mathcal{V}$ , there exists a surjective  $\hat{\mathbf{T}}$ -homomorphism  $e : P_\Sigma \twoheadrightarrow A$  with  $(e')^+ = e \cdot \varphi_\Sigma$ . Then  $F(e) = e'$ , as shown by the commutative diagram below.

$$\begin{array}{ccc}
 T\Sigma & \xrightarrow{\iota_\Sigma} & V\hat{T}\Sigma \\
 \downarrow e' & \searrow V(e')^+ & \downarrow V\varphi_\Sigma \\
 A & \xleftarrow{Ve} & VP_\Sigma
 \end{array}$$

Thus we have the desired connecting arrow  $id : F(e) \rightarrow e'$  in  $(\mathbf{T}\Sigma \downarrow \mathcal{V})$ . The property (ii) is trivially satisfied: since  $(P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$  consists only of surjections, there is at most one morphism  $F(e) \rightarrow e'$  for each  $e$  in  $(P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$ . This shows the finality of  $F$ . Moreover,  $F$  commutes with the projection functors  $\pi$  and  $\pi'$ :

$$\begin{array}{ccc}
 (P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}}) & \xrightarrow{F} & (\mathbf{T}\Sigma \downarrow \mathcal{V}) \\
 \searrow \pi & & \swarrow \pi' \\
 & \mathbf{Alg} \hat{\mathbf{T}} &
 \end{array}$$

The limit of  $\pi$  is  $P_\Sigma$  by Corollary E.4, and the limit of  $\pi'$  is  $\hat{T}_\mathcal{V}\Sigma$  by Remark E.18.1. Thus the finality of  $F$  and the uniqueness of limits implies the existence of an isomorphism

$j_\Sigma : \hat{T}_\Sigma \xrightarrow{\cong} P_\Sigma$  with  $F(e)_\Sigma^+ = e \cdot j_\Sigma$  for all  $e : P_\Sigma \rightarrow A$  in  $(P_\Sigma \downarrow \mathbf{Alg}_f \hat{\mathbf{T}})$ . Consider the diagram below:

The outward triangle commutes by Remark 2.11.2, and all inner parts except for the central triangle commute by the definition of  $j_\Sigma$ , the definition of  $F$  and Remark E.18.3. It follows that the central triangle also commutes, as it commutes when precomposed with the dense map  $\iota_\Sigma$  and postcomposed with the limit projections  $Ve$ . ◀

**Proof of Proposition 4.10.** By Lemma E.20 and E.21 the maps  $\mathcal{V} \mapsto \varphi_\mathcal{V}$  and  $\varphi \mapsto \mathcal{V}_\varphi$  give mutually inverse object maps between the two posets. It remains to show that both constructions are order-preserving.

- (1) Given profinite theories  $\varphi \leq \varphi'$ , we have  $\mathcal{V}_\varphi \subseteq \mathcal{V}_{\varphi'}$  since, for each  $\Sigma \in \mathbb{A}$ , any quotient of  $P_\Sigma$  is also a quotient of  $P'_\Sigma$ .
- (2) Let  $\mathcal{V} \subseteq \mathcal{V}'$  be pseudovarieties. For each  $\Sigma \in \mathbb{A}$  the morphisms  $e_{\mathcal{V}'}^+ : \hat{T}_{\mathcal{V}'}\Sigma \rightarrow A$ , where  $e$  ranges over surjective  $\mathbf{T}$ -homomorphisms  $e : \mathbf{T}\Sigma \rightarrow A$  with  $A \in \mathcal{V}$ , form a compatible family over the diagram defining  $\hat{T}_{\mathcal{V}'}\Sigma$ . Hence there exists a unique morphism  $q : \hat{T}_{\mathcal{V}'}\Sigma \rightarrow \hat{T}_\mathcal{V}\Sigma$  with  $e_{\mathcal{V}'}^+ = e_\mathcal{V}^+ \cdot q$  for all  $e$ . It follows that  $q \cdot \varphi_\Sigma^{\mathcal{V}'} = \varphi_\Sigma^\mathcal{V}$ , because this holds when postcomposed with the limit projections  $e_{\mathcal{V}'}^+$ . Therefore  $\varphi_\mathcal{V} \leq \varphi_{\mathcal{V}'}$ .

**Details for Remark 4.11.** For a class  $E$  of profinite equations over (possibly different) alphabets  $\Sigma \in \mathbb{A}$ , let  $\mathcal{V}[E]$  denote the class of all  $\mathbb{A}$ -generated finite  $\mathbf{T}$ -algebras satisfying all equations in  $E$ . Conversely, for a class  $\mathcal{V}$  of  $\mathbb{A}$ -generated finite  $\mathbf{T}$ -algebras let  $E[\mathcal{V}]$  be the class of all profinite equations over alphabets  $\Sigma \in \mathbb{A}$  satisfied by all algebras in  $\mathcal{V}$ . We claim that  $\mathcal{V}$  forms a pseudovariety iff  $\mathcal{V} = \mathcal{V}[E]$  for some  $E$ .

The “if” direction is an easy verification. For the “only if” direction, let  $\mathcal{V}$  be a pseudovariety of  $\mathbf{T}$ -algebras, and let  $\varphi_\mathcal{V} = (\varphi_\Sigma^\mathcal{V} : \hat{\mathbf{T}}\Sigma \rightarrow \hat{T}_\mathcal{V}\Sigma)_{\Sigma \in \mathbb{A}}$  be the corresponding profinite theory, see Lemma E.19. From the definition of  $\varphi_\mathcal{V}$  it immediately follows that  $\mathcal{V}$  satisfies a profinite equation  $u = v$  over  $\Sigma \in \mathbb{A}$  iff  $u$  and  $v$  are merged by  $\varphi_\Sigma^\mathcal{V}$ . We claim that  $\mathcal{V} = \mathcal{V}[E[\mathcal{V}]]$ . The inclusion  $\subseteq$  is trivial. To prove  $\supseteq$ , let  $A \in \mathcal{V}[E[\mathcal{V}]]$ , i.e.  $A$  satisfies every profinite equation



over  $\mathbb{A}$  that  $\mathcal{V}$  satisfies. Since  $A$  is  $\mathbb{A}$ -generated, there is a surjective  $\mathbf{T}$ -homomorphism  $e : T\mathbb{Z} \twoheadrightarrow A$  with  $\Sigma \in \mathbb{A}$ . By Remark C.3.4,  $A$  is finitely copresentable in  $\mathbf{Alg} \hat{\mathbf{T}}$ , so the  $\hat{\mathbf{T}}$ -homomorphism  $e_{\mathcal{V}}^+ : \hat{T}_{\mathcal{V}}\mathbb{Z} \twoheadrightarrow A$  factors through the limit cone defining  $\hat{T}_{\mathcal{V}}\mathbb{Z}$ ; that is, there exists a surjective  $\mathbf{T}$ -homomorphism  $\bar{e} : T\mathbb{Z} \twoheadrightarrow \bar{A}$  with  $\bar{A} \in \mathcal{V}$  and a  $\mathbf{T}$ -homomorphism  $g : \bar{A} \twoheadrightarrow A$  with  $e_{\mathcal{V}}^+ = g \cdot \bar{e}^+$ . Since  $e_{\mathcal{V}}^+$  is surjective, so is  $g$ . Thus  $A$  is a quotient of  $\bar{A}$  and hence lies in  $\mathcal{V}$ .  $\blacktriangleleft$

## F Details for Section 5

**Proof of Proposition 5.5.** Let  $L : T\mathbb{Z} \rightarrow O_{\mathcal{D}}$  be a recognizable language. By Theorem 3.3 there exists a morphism  $\hat{L} : \hat{T}\mathbb{Z} \rightarrow O_{\mathcal{D}}$  in  $\hat{\mathcal{D}}^S$  with  $L = V\hat{L} \cdot \iota_{\mathbb{Z}}$ .

- (a) Let  $u : (T\mathbb{Z})_s \rightarrow (T\mathbb{Z})_{s'}$  in  $\mathbf{U}_{\Sigma}$ , and take its continuous extension  $\hat{u}$ , see E.6. Then the commutative diagrams below (where  $t \neq s$  in the right diagram), and Theorem 3.3 shows that the language  $u^{-1}L$  is recognizable.

$$\begin{array}{ccc}
 (T\mathbb{Z})_s & \xrightarrow{u} & (T\mathbb{Z})_{s'} \xrightarrow{L} O_{\mathcal{D}} \\
 \downarrow \iota_{\mathbb{Z}} & & \downarrow \iota_{\mathbb{Z}} \\
 V(\hat{T}\mathbb{Z})_s & \xrightarrow{V\hat{u}} & V(\hat{T}\mathbb{Z})_{s'} \xrightarrow{V\hat{L}} O_{\mathcal{D}}
 \end{array}
 \quad
 \begin{array}{ccc}
 (T\mathbb{Z})_t & \xrightarrow{\iota_{\mathbb{Z}}} & V(\hat{T}\mathbb{Z})_t \xrightarrow{V\perp} O_{\mathcal{D}} \\
 & \nearrow u^{-1}L & \\
 & & 
 \end{array}$$

- (b) Let  $g : T\Delta \rightarrow T\mathbb{Z}$  be a  $\mathbf{T}$ -homomorphism and  $\hat{g}$  its continuous extension, see Lemma C.4. Then the language  $g^{-1}L = L \cdot g$  is recognizable by the commutative diagram below and Theorem 3.3.

$$\begin{array}{ccc}
 T\Delta & \xrightarrow{g} & T\mathbb{Z} \xrightarrow{L} O_{\mathcal{D}} \\
 \downarrow \iota_{\Delta} & & \downarrow \iota_{\mathbb{Z}} \\
 V\hat{T}\Delta & \xrightarrow{V\hat{g}} & V\hat{T}\mathbb{Z} \xrightarrow{V\hat{L}} O_{\mathcal{D}}
 \end{array}$$

► **Remark F.1.** 1. For each  $\Sigma \in \mathbf{Set}_f^S$  and each sort  $s$  we have

$$\begin{aligned}
 |P(\hat{T}\mathbb{Z})_s| &\cong \mathcal{C}(\mathbf{1}, P(\hat{T}\mathbb{Z})_s) \\
 &\cong \hat{\mathcal{D}}((\hat{T}\mathbb{Z})_s, O_{\mathcal{D}}) \\
 &\cong \{ (T\mathbb{Z})_s \xrightarrow{L_s} O_{\mathcal{D}} : L \in \mathbf{Reg}(\Sigma) \}.
 \end{aligned}$$

The last bijection is given by  $\hat{f} \mapsto V\hat{f} \cdot \iota_{\mathbb{Z}}$ . Indeed, observe that for each recognizable language  $L : T\mathbb{Z} \rightarrow O_{\mathcal{D}}$ , the language  $L' : T\mathbb{Z} \rightarrow O_{\mathcal{D}}$  with  $L'_s = L_s$  and  $L'_t = V\perp \cdot \iota_{\mathbb{Z}}$  for  $t \neq s$  is also recognizable (by the same  $\mathbf{T}$ -homomorphism, using the naturality of  $\perp$ ). From this and Theorem 3.3 the bijection immediately follows.

From now on let us assume that  $P(\hat{T}\mathbb{Z})_s$  is carried by  $\{L_s : L \in \mathbf{Rec}(\Sigma)\}$ . With this identification, the isomorphism  $\mathbf{Rec}(\Sigma) \cong \prod_s P(\hat{T}\mathbb{Z})_s$  of Remark 3.4 maps a language  $L : T\mathbb{Z} \rightarrow O_{\mathcal{D}}$  to the tuple  $((T\mathbb{Z})_s \xrightarrow{L_s} O_{\mathcal{D}})_{s \in S}$  of its components.

2. For any subobject  $V_\Sigma \subseteq \text{Rec}(\Sigma)$  and any sort  $s$ , let  $m_s: (V'_\Sigma)_s \rightarrow P(\hat{T}\Sigma)_s$  be the subobject of  $P(\hat{T}\Sigma)_s$  making the following diagram commute:

$$\begin{array}{ccc} V_\Sigma & \xrightarrow{\subseteq} & \text{Rec}(\Sigma) \\ \downarrow e_s & & \downarrow \cong \\ & \prod_s P(\hat{T}\Sigma)_s & \\ & \downarrow \pi_s & \\ (V'_\Sigma)_s & \xrightarrow{m_s} & P(\hat{T}\Sigma)_s \end{array}$$

By point 1 above,  $(V'_\Sigma)_s$  is (up to isomorphism) carried by  $\{(T\Sigma)_s \xrightarrow{L_s} O_{\mathcal{D}}) : L \in V_\Sigma\}$ . For  $e = \langle e_s \rangle_{s \in S}$  we get the following diagram:

$$\begin{array}{ccc} V_\Sigma & \xrightarrow{\subseteq} & \text{Rec}(\Sigma) \\ \downarrow e & & \downarrow \cong \\ \prod_s (V'_\Sigma)_s & \xrightarrow{\prod_s m_s} & \prod_s P(\hat{T}\Sigma)_s \end{array}$$

Clearly  $e$  is monic. The subobject  $V_\Sigma$  is called *admissible* if  $e$  is also surjective (i.e. an isomorphism), cf. Remark 5.6. This means precisely that  $V_\Sigma$  is closed under *diagonals*: for any  $S$ -indexed family  $L^s$  ( $s \in S$ ) of languages in  $V_\Sigma$ , the *diagonal language*  $L^*: T\Sigma \rightarrow O_{\mathcal{D}}$  with  $L_s^* = L_s^s$  lies in  $V_\Sigma$ .

3. Every subobject  $V_\Sigma \subseteq \text{Rec}(\Sigma)$  contains the “empty language”, i.e. the language with  $V \perp \cdot \iota_\Sigma: (T\Sigma)_t \rightarrow O_{\mathcal{D}}$  in each sort  $t$ . Indeed, by the definition of  $\perp$  (the dual of the natural transformation choosing a constant, see Remark 5.1) this language is precisely the constant in  $\text{Rec}(\Sigma) \cong \prod_s P(\hat{T}\Sigma)_s$ , and every subobject of  $\text{Rec}(\Sigma)$  contains the constant.

**Details for Remark 5.8.** Suppose that  $\mathbf{U}_\Sigma$  contains all identity morphisms, and let  $\mathcal{C}$  be one of the varieties of Example 2.2 and 2.3. We claim that any subobject  $V_\Sigma \subseteq \text{Rec}(\Sigma)$  closed under derivatives is admissible, i.e. closed under diagonals, see Remark F.1.2. Thus suppose that  $L^s$  ( $s \in S$ ) is an  $S$ -indexed family in  $V_\Sigma$ . Since  $V_\Sigma$  is closed under derivatives and  $\mathbf{U}_\Sigma$  contains all identity morphisms, the language  $(id_{(T\Sigma)_s})^{-1}L^s$  lies in  $V_\Sigma$  for each  $s$ . Recall that  $\perp$  has been chosen as the zero map, see Remark 5.1. Therefore this derivative agrees with  $L^s$  in sort  $s$ , and is empty in all other sorts. Finally, observe that for  $\mathcal{C} = \mathbf{BA}, \mathbf{DL}_{01}, \mathbf{JSL}_0$ , the set  $V_\Sigma$  is closed under union, since  $\text{Rec}(\Sigma) \rightarrow \prod_s O_{\mathcal{C}}^{|T\Sigma|_s}$  by Remark 3.4. Thus the diagonal language  $L^* = \bigcup_s (id_{(T\Sigma)_s})^{-1}L^s$  lies in  $V_\Sigma$ . Analogously for  $\mathcal{C} = \mathbf{Vec}_K$  where  $V_\Sigma$ , being a subspace of  $\text{Rec}(\Sigma)$ , is closed under taking sums of languages.  $\blacktriangleleft$

► **Lemma F.2.** Let  $\mathbf{U}_\Sigma$  be a unary presentation of  $\mathbf{T}$  over  $\Sigma$ , and let  $V_\Sigma \subseteq \text{Rec}(\Sigma)$  be an admissible subobject of  $\text{Rec}(\Sigma)$ , represented by subobjects  $m_t: (V'_\Sigma)_t \rightarrow P(\hat{T}\Sigma)_t$  ( $t \in S$ ). For any  $u: (T\Sigma)_s \rightarrow (T\Sigma)_{s'}$  in  $\mathbf{U}_\Sigma$  the following statements are equivalent:

- (i)  $u^{-1}L \in V_\Sigma$  for all  $L \in V_\Sigma$ .
- (ii) There exists a morphism  $u'$  making (10) commute, where  $\hat{u}: (\hat{T}\Sigma)_s \rightarrow (\hat{T}\Sigma)_{s'}$  is the continuous extension of  $u$  (see Lemma E.6):

$$\begin{array}{ccc} (V'_\Sigma)_{s'} & \xrightarrow{u'} & (V'_\Sigma)_s \\ m_{s'} \downarrow & & \downarrow m_s \\ P(\hat{T}\Sigma)_{s'} & \xrightarrow{P\hat{u}} & P(\hat{T}\Sigma)_s. \end{array} \tag{10}$$

In particular,  $V_\Sigma$  is a local variety (w.r.t.  $\mathbb{U}_\Sigma$ ) iff  $u'$  with (10) exists for every  $u \in \mathbb{U}_\Sigma$ .

**Proof.** Recall from Remark F.1 that  $P(\hat{T}\Sigma)_t$  is, up to isomorphism, carried by the set  $\{L_t : L \in \text{Rec}(\Sigma)\}$ , and  $(V'_\Sigma)_t$  by the subset  $\{L_t : L \in V_\Sigma\}$ . From the definition of  $\hat{u}$  it follows that  $P\hat{u}$  takes an element  $L_{s'}$  of  $P(\hat{T}\Sigma)_{s'}$  to  $L_{s'} \cdot u$ . Thus (ii) is equivalent to the statement that  $L_{s'} \cdot u \in (V'_\Sigma)_s$  for all  $L \in V_\Sigma$ . From this observation the implication (i) $\Rightarrow$ (ii) follows immediately, since  $(u^{-1}L)_s = L_{s'} \cdot u$ .

Conversely, suppose that (ii) holds, and let  $L \in V_\Sigma$ . By the above argument, we have  $L_{s'} \cdot u \in (V'_\Sigma)_s$ . Moreover, by Remark F.1.3 the “empty language” with  $V \perp \cdot \iota_\Sigma$  in each sort lies in  $V_\Sigma$ . The admissibility of  $V_\Sigma$  (i.e. closure under diagonals, see Remark F.1) thus implies that the language with  $L_{s'} \cdot u$  in sort  $s$  and  $V \perp \cdot \iota_\Sigma$  in all sorts  $t \neq s$  lies in  $V_\Sigma$ . But this is precisely the derivative  $u^{-1}L$ , which proves (ii) $\Rightarrow$ (i).  $\blacktriangleleft$

► **Lemma F.3.** For  $\Sigma, \Delta \in \text{Set}_f^S$  let  $V_\Sigma \subseteq \text{Rec}(\Sigma)$  and  $V_\Delta \subseteq \text{Rec}(\Delta)$  be admissible subobjects, represented by  $m_s^\Sigma : (V'_\Sigma)_s \rightarrow P(\hat{T}\Sigma)_s$  and  $m_s^\Delta : (V'_\Delta)_s \rightarrow P(\hat{T}\Delta)_s$  ( $s \in S$ ). Then for any  $\mathbf{T}$ -homomorphism  $g : \mathbf{T}\Delta \rightarrow \mathbf{T}\Sigma$ , the following statements are equivalent:

- (i)  $g^{-1}L \in V_\Delta$  for all  $L \in V_\Sigma$ .
- (ii) There is a morphism  $g' : V'_\Sigma \rightarrow V'_\Delta$  in  $\mathcal{C}^S$  making the following square commute for any sort  $s$ , where  $\hat{g} : \hat{\mathbf{T}}\Delta \rightarrow \hat{\mathbf{T}}\Sigma$  is the continuous extension of  $g$  (see Lemma C.4).

$$\begin{array}{ccc} (V'_\Sigma)_s & \xrightarrow{g'_s} & (V'_\Delta)_s \\ m_s^\Sigma \downarrow & & \downarrow m_s^\Delta \\ P(\hat{T}\Sigma)_s & \xrightarrow{P\hat{g}_s} & P(\hat{T}\Delta)_s \end{array} \quad (11)$$

**Proof.** Again we use that  $P(\hat{T}\Sigma)_s$  is, up to isomorphism, carried by the set  $\{L_s : L \in \text{Rec}(\Sigma)\}$ , and  $(V'_\Sigma)_s$  by the subset  $\{L_s : L \in V_\Sigma\}$ . Analogously for  $P(\hat{T}\Delta)_s$  and  $(V'_\Delta)_s$ . From the definition of  $\hat{g}$  it follows that  $P\hat{g}_s$  takes an element  $L_s$  of  $P(\hat{T}\Sigma)_s$  to  $L_s \cdot g_s$ . Thus (ii) is equivalent to the statement that  $L_s \cdot g_s \in (V'_\Delta)_s$  for all  $L \in V_\Sigma$  and all sorts  $s$ . From this the implication of (i) $\Rightarrow$ (ii) is obvious. Conversely, suppose that (ii) holds, and let  $L \in V_\Sigma$ . By the above argument, we have  $L_s \cdot g_s \in (V'_\Delta)_s$  for all  $s$ . By admissability of  $V_\Delta$ , this implies that  $g^{-1}L = L \cdot g$  lies in  $V_\Delta$ , i.e. (ii) $\Rightarrow$ (i) holds.  $\blacktriangleleft$

**Proof of Theorem 5.9.** We first prove the local variety theorem. Let  $V_\Sigma \subseteq \text{Rec}(\Sigma)$  be an admissible subobject, represented by a subobject

$$m = ( (V'_\Sigma)_s \xrightarrow{m_s} P(\hat{T}\Sigma)_s )_{s \in S}$$

in  $\mathcal{C}^S$ . From Lemma F.2 and E.8, it follows that  $V_\Sigma$  forms a local variety of languages iff the dual quotient

$$( (\hat{T}\Sigma)_s \xrightarrow{\cong} P^{-1}P(\hat{T}\Sigma)_s \xrightarrow{P^{-1}m_s} P^{-1}(V'_\Sigma)_s )_{s \in S}$$

in  $\hat{\mathcal{D}}^S$  carries a  $\Sigma$ -generated profinite  $\hat{\mathbf{T}}$ -algebra. Then Proposition 4.3 gives the isomorphism between local varieties of languages over  $\Sigma$  and local pseudovarieties of  $\Sigma$ -generated  $\mathbf{T}$ -algebras.

For the non-local variety theorem, observe further that by Lemma F.3, a family  $(V_\Sigma \subseteq \text{Rec}(\Sigma))_{\Sigma \in \mathbb{A}}$  of local varieties forms a variety of languages (i.e., is closed under preimages) iff the dual family of  $\Sigma$ -generated profinite  $\hat{\mathbf{T}}$ -algebras forms a profinite theory. Then Proposition 4.10 gives the isomorphism between varieties of languages and pseudovarieties of  $\mathbf{T}$ -algebras.  $\blacktriangleleft$

**Details for Remark 5.10.** Let  $\mathbf{C}$  be a family associating to each pair  $(\Sigma, \Delta) \in \mathbb{A}^2$  a set  $\mathbf{C}(\Delta, \Sigma)$  of  $\mathbf{T}$ -homomorphisms from  $\mathbf{T}\Delta$  to  $\mathbf{T}\Sigma$ . A  $\mathbf{C}$ -variety of languages is given as in Definition 5.7.2, but with  $g$  restricted to elements of  $\mathbf{C}$ . Similarly, a *profinite  $\mathbf{C}$ -theory* is given as in Definition 4.9, but with  $g$  again restricted to  $\mathbf{C}$ . Then we get

► **Theorem F.4** (Straubing Theorem for  $\mathbf{C}$ -varieties). *The poset of  $\mathbf{C}$ -varieties of languages is isomorphic to the poset of profinite  $\mathbf{C}$ -theories.*

This follows via duality from Lemma F.3 and Lemma E.8, in complete analogy to the proof of Theorem 5.9. For the monad  $\mathbf{T} = \mathbf{T}_*$  on **Set**, this theorem is due to Straubing [35]. ◀

## G Details for Section 6

We provide some details for the case of finite words. Let  $\mathcal{D}$  be a commutative variety of algebras or ordered algebras. Then  $(\mathcal{D}, \otimes, \mathbb{1})$  is a symmetric monoidal closed category w.r.t. the usual tensor product  $\otimes$  (representing bimorphisms), see [7]. By a  $\mathcal{D}$ -monoid we mean an object  $D \in \mathcal{D}$  equipped with a monoid structure  $(|D|, \bullet, 1)$  on the underlying set such that the multiplication  $\bullet : |D| \times |D| \rightarrow |D|$  is a bimorphism of  $\mathcal{D}$ ; that is, for every  $x \in |D|$  the maps

$$x \bullet - : |D| \rightarrow |D| \quad \text{and} \quad - \bullet x : |D| \rightarrow |D|$$

carry endomorphisms of  $D$ . Equivalently, a  $\mathcal{D}$ -monoid is a monoid object in  $(\mathcal{D}, \otimes, \mathbb{1})$ . A *morphism* of  $\mathcal{D}$ -monoids is a morphism in  $\mathcal{D}$  that also preserves the monoid structure.

Consider the following diagram of left and right adjoints, where **Mon** and **Mon**( $\mathcal{D}$ ) are the categories of monoids and  $\mathcal{D}$ -monoids and  $U, U'$  are the forgetful functors. Both the outer and the inner square commute.

$$\begin{array}{ccc}
 \mathcal{D} & \xleftarrow[U']{\tau} & \mathbf{Mon}(\mathcal{D}) \\
 \downarrow |-| \quad \uparrow \Psi & \xleftarrow[F']{(-)^*} & \downarrow F \quad \uparrow U \\
 \mathbf{Set} & \xleftarrow[\perp]{|-|} & \mathbf{Mon}
 \end{array}$$

The left adjoint  $F$  sends a monoid  $M = (M, \cdot, e)$  to the  $\mathcal{D}$ -monoid  $FM = (\Psi|M|, \bullet, e)$ , where  $\bullet : \Psi|M| \times \Psi|M| \rightarrow \Psi|M|$  is the unique bimorphism in  $\mathcal{D}$  that extends the multiplication  $\cdot : |M| \times |M| \rightarrow |M|$ , and it sends a monoid morphism  $h : M \rightarrow M'$  to  $\Psi|h| : \Psi|M| \rightarrow \Psi|M'|$ . This implies that the free  $\mathcal{D}$ -monoid on a free object  $\Sigma$  in  $\mathcal{D}$  is given by

$$F'\Sigma = F'\Psi\Sigma = F\Sigma^* = (\Psi\Sigma^*, \bullet, \varepsilon),$$

where  $\varepsilon$  is the empty word and  $\bullet$  extends the concatenation of words.

Let  $\mathbf{T}_M$  be the monad on  $\mathcal{D}$  associated to the adjunction  $F' \dashv U'$ , i.e. constructing free  $\mathcal{D}$ -monoids on objects of  $\mathcal{D}$ . Clearly  $U'$  is monadic, and thus  $\mathbf{Alg}(\mathbf{T}_M) \cong \mathbf{Mon}(\mathcal{D})$ . A language  $L : \mathbf{T}_M\Sigma \rightarrow O_{\mathcal{D}}$  in  $\mathcal{D}$  corresponds (via the adjunction  $\Psi \dashv |-| : \mathcal{D} \rightarrow \mathbf{Set}$ ) to a function  $L' : \Sigma^* \rightarrow |O_{\mathcal{D}}|$ .

► **Lemma G.1.**  *$L$  is  $\mathbf{T}_M$ -recognizable iff  $L'$  is regular, i.e. computed by some finite Moore automaton with output set  $|O_{\mathcal{D}}|$ .*

**Proof.** For  $\mathcal{D} = \mathbf{Set}$  with  $O_{\mathbf{Set}} = \{0, 1\}$ , this is the well-known equivalence of regular and monoid-recognizable languages, see e.g. [26]. Now let  $\mathcal{D}$  be any commutative variety. If  $L$  is recognizable, there exists a  $\mathcal{D}$ -monoid morphism  $h : \Psi\Sigma^* \rightarrow D$ , where  $D$  is finite, and a morphism  $p : D \rightarrow O_{\mathcal{D}}$  in  $\mathcal{D}$  with  $L = p \cdot h$ . Then  $h$  restricts to a monoid morphism

$$h' = (\Sigma^* \rightarrow U\Psi\Sigma^* \xrightarrow{U_h} UD)$$

that recognizes  $L'$  via  $|p|$ . Thus  $L'$  is regular.

Conversely, suppose that  $L'$  is regular. Then  $L'$  is monoid-recognizable (in  $\mathbf{Set}$ ), so there exists a monoid morphism  $h : \Sigma^* \rightarrow M$ , where  $M$  is a finite monoid, and a function  $p : M \rightarrow |O_{\mathcal{D}}|$  such that  $L' = p \cdot h$ . Let  $p' : \Psi M \rightarrow O_{\mathcal{D}}$  in  $\mathcal{D}$  be the adjoint transpose of  $p$  (via the adjunction  $\Psi \dashv |-| : \mathcal{D} \rightarrow \mathbf{Set}$ ). Then  $\Psi h : \Psi\Sigma^* \rightarrow \Psi M$  is a  $\mathcal{D}$ -monoid morphism that recognizes  $L$  via  $p'$ .  $\blacktriangleleft$